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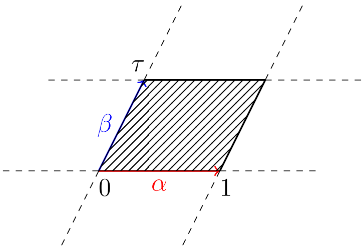
Elliptic Multiple Zeta Values



$$\int_{\alpha} \bigwedge^{n} \omega_{\mathrm{KZB}} = \sum_{k_1,\dots,k_n \geq 0} I^{\mathrm{A}}(k_1,\dots,k_n;\tau) X_1^{k_1-1} \dots X_n^{k_n-1}$$

$$(2\pi i)^2 I^{\mathrm{A}}(0,1,0,0;\tau) = -3\zeta(3) + 6q + \frac{27}{4}q^2 + \frac{56}{9}q^3 + \dots$$

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Introduction

The topic of this thesis are elliptic multiple zeta values, which are an elliptic analogue of the well-studied multiple zeta values. Elliptic multiple zeta values are given by convergent power series in the variable $q = e^{2\pi i\tau}$ (where τ denotes the canonical coordinate on the upper half-plane $\mathbb{H}$), whose coefficients are linear combinations of multiple zeta values. In particular, they are holomorphic functions on the upper half-plane, which degenerate to multiple zeta values at the cusp $i\infty$ of $\mathbb{H}$. As there is a structural parallelism between multiple zeta values and elliptic multiple zeta values, we begin by reviewing some facets of the theory of multiple zeta values before describing their elliptic analogues.

Multiple zeta values are real numbers, given for positive integers $k_1, \ldots, k_{n-1} \geq 1$ and $k_n \geq 2$ by the sum

$$\zeta(k_1, \ldots, k_n) = \sum_{0<m_1<\ldots<m_n} \frac{1}{m_1^{k_1} \ldots m_n^{k_n}}. \tag{0.1}$$

They are generalizations of the special values of the Riemann zeta function, and are known to arise in a variety of different areas in mathematics and mathematical physics, such as knot theory [53], quantum groups [30], the theory of motives [40], perturbative quantum field theory [12], superstring theory [67], and others. In all of these contexts, a central object of study is the $\mathbb{Q}$-algebra $\mathcal{Z}$ generated by the multiple zeta values: To describe $\mathcal{Z}$ as precisely as possible is one of the main topics of research in multiple zeta value theory.

One way to study multiple zeta values is by realizing them as periods of certain algebro-geometric objects [15, 29, 41]. This point of view has lead to a deep relation between multiple zeta values and mixed Tate motives over $\mathbb{Z}$, which puts strong constraints on the algebraic structure of $\mathcal{Z}$ [16, 40, 74]. An important role is played by the general notion of *homotopy invariant iterated integral* on a smooth manifold, which has been developed extensively by Chen, and later by Hain in the context of rational homotopy theory of algebraic varieties [25, 46]. The upshot is that multiple

zeta values can be written as homotopy invariant iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which relates the study of the algebra $\mathcal{Z}$ to the geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [26].
By a result of Kontsevich, multiple zeta values also occur as coefficients of the *Drinfeld associator* [30], a formal power series, which describes the monodromy of the *Knizhnik-Zamolodchikov (KZ) equation* known from conformal field theory [52]. The Drinfeld associator satisfies certain functional identities, which yield a family of algebraic relations between multiple zeta values, related to representations of braid Lie algebras. It is conjectured that these *associator relations* exhaust all algebraic relations between multiple zeta values (cf. [2], §25.4) for the precise conjecture and [36] for some recent progress).
The interpretation of multiple zeta values as periods on the one hand, and as the monodromy of the KZ equation on the other hand has lead to several far-reaching conjectures about multiple zeta values, some of which are presented in Chapter 1 (cf. [2], §25.4 for a more detailed account). Although partial results towards a resolution of these conjectures have been obtained, the algebraic structure of multiple zeta values still awaits a definitive description.

On the other hand, *elliptic multiple zeta values* have been introduced in [32] as an extension of the notion of multiple zeta value to elliptic curves. They are defined by homotopy invariant iterated integrals on a once-punctured complex elliptic curve $E_\tau^\times = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \setminus \{0\}$. Such iterated integrals have been studied in the context of the de Rham homotopy theory of $E_\tau^\times$ [23], as well as in the theory of universal mixed elliptic motives [45]. In both contexts, they give rise to (multiple) elliptic polylogarithms, which were first introduced by Bloch [9] in a special case, and later extended and studied intensively by many people [6, 8, 23, 51, 54, 55, 80, 81]. Elliptic multiple zeta values are then obtained by evaluating these multiple elliptic polylogarithms along the canonical paths α, β on $E_\tau^\times$, which correspond to the two usual homology cycles on $E_\tau^\times$.
A second representation of elliptic multiple zeta values is as coefficients of the elliptic Knizhnik-Zamolodchikov-Bernard (KZB) associator [31]. In fact, this is the approach to elliptic multiple zeta values, which is used in this thesis. The elliptic KZB associator is essentially a triple $(\Phi_{\mathrm{KZ}}, \underline{A}(\tau), \underline{B}(\tau))$ of formal power series in non-commuting variables x_0 and x_1, which describes the regularized monodromy of the elliptic KZB equation [24, 43, 55] along the paths α and β on $E_\tau^\times$ alluded to above. Here, Φ_{KZ} denotes the Drinfeld associator [30], and the series $\underline{A}(\tau)$ and $\underline{B}(\tau)$ are obtained by iterated integration of the elliptic KZB equation along the paths

α resp. β. The relation between the elliptic KZB associator and multiple elliptic polylogarithms now relies on a theorem of Brown and Levin [23], which states that every homotopy invariant iterated integral on $E_\tau^\times$ can be obtained from the elliptic KZB equation.

The classical Kronecker series $F_\tau(\xi,\alpha) = \frac{\theta'_\tau(0)\theta_\tau(\xi+\alpha)}{\theta_\tau(\xi)\theta_\tau(\alpha)}$ [78, 82], where $\theta_\tau(\xi)$ denotes the odd Jacobi theta function, features prominently in the definition of the elliptic KZB equation. Thus, the elliptic KZB associator is related to classical elliptic functions. This relation, which has no analogue for multiple zeta values, is one of the key features of elliptic multiple zeta values.

We now return to elliptic multiple zeta values. The coefficients of the power series $\underline{A}(\tau)$ and $\underline{B}(\tau)$ span two $\mathbb{Q}$-algebras

$$\mathcal{EZ}^{\mathrm{A}} = \mathrm{Span}_{\mathbb{Q}}\{\underline{A}(\tau)_{|w} \mid w \in \langle x_0, x_1\rangle\}, \tag{0.2}$$

$$\mathcal{EZ}^{\mathrm{B}} = \mathrm{Span}_{\mathbb{Q}}\{\underline{B}(\tau)_{|w} \mid w \in \langle x_0, x_1\rangle\}, \tag{0.3}$$

where $\langle x_0, x_1\rangle$ denotes the set of all monomials in the variables x_0 and x_1. We will call $\mathcal{EZ}^{\mathrm{A}}$ the algebra of *A-elliptic multiple zeta values*, and likewise $\mathcal{EZ}^{\mathrm{B}}$ the algebra of *B-elliptic multiple zeta values.*

Goal. *Understand the structure of the algebras $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$.*

In order to pursue this goal, the plan is to use the theory of multiple zeta values as a guide. In fact, we will see that many algebraic properties of multiple zeta values have analogues for elliptic multiple zeta values. In particular, the structure of the $\mathbb{Q}$-algebras $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$ is reminiscent of the structure of the $\mathbb{Q}$-algebra $\mathcal{Z}$ of multiple zeta values.

Decomposition of elliptic multiple zeta values

A key result towards understanding the algebraic structure of elliptic multiple zeta values is the fact that they satisfy a linear differential equation on the upper half-plane $\mathbb{H}$ [31]. This differential equation identifies elliptic multiple zeta values as special linear combinations of iterated integrals of Eisenstein series, which will be called *iterated Eisenstein integrals* for short

$$\mathcal{E}(2k_1,\dots,2k_n;\tau) = \int_\tau^{i\infty} E_{2k_1}(\tau_1)\mathrm{d}\tau_1 \dots E_{2k_n}(\tau_n)\mathrm{d}\tau_n, \quad k_1,\dots,k_n \geq 0, \tag{0.4}$$

where for $k \geq 1$, $E_{2k}(\tau) = \frac{(2k-1)!}{2(2\pi i)^{2k}} \sum_{(m,n)\in\mathbb{Z}^2,\,(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}}$ denotes the holomorphic Eisenstein series, and we set $E_0 = -1$. These iterated Eisenstein integrals are

a special case of the more general concept of iterated integrals of modular forms, or *iterated Shimura integrals*, whose study was initiated by Manin [57] (in the case of cusp forms) and was extended recently by Brown [20].
Together with a suitable asymptotic condition at the cusp $i\infty$ given in terms of multiple zeta values [31, 32], the differential equation yields a canonical decomposition of elliptic multiple zeta values into linear combinations of iterated Eisenstein integrals and multiple zeta values. Inspired by the appearance of elliptic multiple zeta values in superstring theory [13], this decomposition has been studied first in [14].
The decomposition of elliptic multiple zeta values into iterated Eisenstein integrals greatly clarifies the algebraic structure of elliptic multiple zeta values. In order to see this, let $\langle\mathcal{E}\rangle_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space spanned by the iterated Eisenstein integrals. The shuffle product formula for iterated integrals [65] implies that $\langle\mathcal{E}\rangle_{\mathbb{Q}}$ is in fact a $\mathbb{Q}$-algebra. We also denote by $T(\mathbf{e})^\vee$ the graded dual of the tensor algebra $T(\mathbf{e})$ on the free $\mathbb{Q}$-vector space spanned by the set $\mathbf{e} = \{\mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_4, \dots\}$. Elements of $T(\mathbf{e})^\vee$ can thus be identified with linear combinations of words $\mathbf{e}^\vee_{2k_1} \dots \mathbf{e}^\vee_{2k_n}$ in the dual letters $\mathbf{e}^\vee_{2k}$, i.e. $\mathbf{e}^\vee_{2k}(\mathbf{e}_{2l}) = \delta_{k,l}$, and the product is the shuffle product (cf. [66], I.4).

Theorem 1 (cf. Theorem 5.1.3)**.** *The $\mathbb{Q}$-algebra $\langle\mathcal{E}\rangle_{\mathbb{Q}}$ is a free shuffle algebra. More precisely, the morphism*

$$\begin{aligned} T(\mathbf{e})^\vee &\to \langle\mathcal{E}\rangle_{\mathbb{Q}} \\ \mathbf{e}^\vee_{2k_1} \dots \mathbf{e}^\vee_{2k_n} &\mapsto \mathcal{E}(2k_1, \dots, 2k_n; \tau) \end{aligned} \tag{0.5}$$

is an isomorphism of $\mathbb{Q}$-algebras.

The theorem follows from linear independence of iterated Eisenstein integrals, proved by the author in [58]. We note that by a result of Radford [63], it implies that $\langle\mathcal{E}\rangle_{\mathbb{Q}}$ is a polynomial algebra in infinitely many variables, given by the Lyndon words on the alphabet $\mathbf{e}$ ([66], Chapter 5).
We now return to elliptic multiple zeta values. It turns out that the decomposition into linear combinations of iterated Eisenstein series is simplified if instead of the algebras $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$, which were defined as the linear spans of the coefficients of the series $\underline{A}(\tau)$, $\underline{B}(\tau)$, we consider the algebras

$$\overline{\mathcal{EZ}}^{\mathrm{A}} = \{(2\pi i)^{-d(w)}\underline{A}(\tau)_w \,|\, w \in \langle x_0, x_1\rangle\}, \tag{0.6}$$

$$\overline{\mathcal{EZ}}^{\mathrm{B}} = \{(2\pi i)^{-d(w)}\underline{B}(\tau)_w \,|\, w \in \langle x_0, x_1\rangle\}, \tag{0.7}$$

where $d(w) = \deg_{x_0}(w) - \deg_{x_1}(w)$. This has the effect of clearing powers of $2\pi i$ from the denominators. Now combining the expansion of elliptic multiple zeta values into iterated Eisenstein integrals with the isomorphism (0.5), we obtain embeddings

$$\iota_A : \overline{\mathcal{EZ}}^{\mathrm{A}} \hookrightarrow T(\mathbf{e})^{\vee} \otimes \mathcal{Z}[2\pi i], \tag{0.8}$$

$$\iota_B : \overline{\mathcal{EZ}}^{\mathrm{B}} \hookrightarrow T(\mathbf{e})^{\vee} \otimes \mathcal{Z} \tag{0.9}$$

of $\mathbb{Q}$-algebras[1]. The definition of the morphisms ι_A and ι_B above are completely canonical, and an explicit construction was given in [14]. Describing the image of this map is equivalent to giving all linear combinations of iterated Eisenstein integrals and multiple zeta values, which occur as elliptic multiple zeta values.
In this direction, it is proved in this thesis that both ι_A and ι_B factor through a smaller subalgebra as follows. Let $\mathcal{L}$ be the free Lie algebra in two generators x_0, x_1. There exists a distinguished Lie subalgebra $\mathfrak{u}^{\mathrm{geom}} \subset \mathrm{Der}(\mathcal{L})$ of the algebra of derivations on $\mathcal{L}$, which is generated by derivations ε_{2k}, for $k \geq 0$ [60, 75]. These derivations are "geometric", in the sense that they describe the universal $\mathrm{SL}_2(\mathbb{Z})$-monodromy on the once-punctured elliptic curve $E_\tau^{\times}$ [45].
Since the universal enveloping algebra $U(\mathfrak{u}^{\mathrm{geom}})$ is generated by one element in each even degree, the universal property of the shuffle algebra $T(\mathbf{e})^{\vee}$ yields an embedding

$$U(\mathfrak{u}^{\mathrm{geom}})^{\vee} \hookrightarrow T(\mathbf{e})^{\vee} \tag{0.10}$$

of the graded dual of $U(\mathfrak{u}^{\mathrm{geom}})$ into $T(\mathbf{e})^{\vee}$. As the generators ε_{2k} are not free [45, 61], the image of (0.10) is contained in a proper subspace of $T(\mathbf{e})^{\vee}$, which is cut out by equations, which are orthogonal to the relations holding in $U(\mathfrak{u}^{\mathrm{geom}})$ [14].

Theorem 2 (cf. Theorem 5.3.1)**.** *Both embeddings* (0.8) *and* (0.9) *factor through* $U(\mathfrak{u}^{\mathrm{geom}})^{\vee}$*, i.e. we have*

$$\iota_A : \overline{\mathcal{EZ}}^{\mathrm{A}} \hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^{\vee} \otimes \mathcal{Z}[2\pi i], \tag{0.11}$$

$$\iota_B : \overline{\mathcal{EZ}}^{\mathrm{B}} \hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^{\vee} \otimes \mathcal{Z}. \tag{0.12}$$

This result is interesting because of the relation between the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ and modular forms for $\mathrm{SL}_2(\mathbb{Z})$. It is known that non-trivial relations in the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ are related to the existence of period polynomials for modular forms [45, 61].

[1]In contrast, the algebras $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$ embed only into $T(\mathbf{e})^{\vee} \otimes \mathcal{Z}[(2\pi i)^{-1}]$

In the graded dual $U(\mathfrak{u}^{\mathrm{geom}})^\vee$, these relations become constraints on the image of the embedding $U(\mathfrak{u}^{\mathrm{geom}})^\vee \hookrightarrow \langle \mathcal{E} \rangle_{\mathbb{Q}}$. More concisely, the linear combinations of iterated Eisenstein integrals which occur in the decomposition of elliptic multiple zeta values are constrained by modular forms.

Although at the moment, we cannot describe the image of ι_A and ι_B completely, we have obtained some partial results. In order to describe these, we introduce the length decomposition of an elliptic multiple zeta value. For a fixed non-negative integer $n \geq 0$, we can, by the above, rewrite an elliptic multiple zeta value as a linear combination of iterated Eisenstein integrals, and then project onto the iterated Eisenstein integrals of length n. For an elliptic multiple zeta value, the highest such n for which there is a non-zero contribution is called the *highest length component* of the elliptic multiple zeta value. One can show that the highest length component is always a $\mathbb{Q}$-linear combination of iterated Eisenstein integrals, as opposed to a general $\mathcal{Z}[2\pi i]$-linear combination. Likewise, there is a notion of *lowest length component*, which is contained in $\mathcal{Z}[2\pi i]$ (even in $\mathcal{Z}$ for B-elliptic multiple zeta values). Denote by

$$\iota_A^{\mathrm{geom}} : \overline{\mathcal{E}\mathcal{Z}}^{\mathrm{A}} \to U(\mathfrak{u}^{\mathrm{geom}})^\vee, \tag{0.13}$$

$$\iota_B^{\mathrm{geom}} : \overline{\mathcal{E}\mathcal{Z}}^{\mathrm{B}} \to U(\mathfrak{u}^{\mathrm{geom}})^\vee \tag{0.14}$$

the projections onto the highest length component and likewise by

$$\iota_A^{\zeta} : \overline{\mathcal{E}\mathcal{Z}}^{\mathrm{A}} \to \mathcal{Z}[2\pi i], \tag{0.15}$$

$$\iota_B^{\zeta} : \overline{\mathcal{E}\mathcal{Z}}^{\mathrm{B}} \to \mathcal{Z} \tag{0.16}$$

the projections onto the lowest length component.

Theorem 3 (cf. Theorem 5.4.10, Theorem 5.4.13 and Theorem 5.4.2)**.** (i) *The morphism ι_B^{geom} is surjective.*

(ii) *The image of ι_A^{geom} is contained in a proper subspace of $U(\mathfrak{u}^{\mathrm{geom}})^\vee$, the "Fourier subspace" (cf. Definition 5.4.11) corresponding to those iterated Eisenstein integrals which have a Fourier expansion.*

(iii) *The morphism ι_B^{ζ} is surjective, while the image of the morphism ι_A^{ζ} is the subspace*

$$\mathbb{Q} + 2\pi i \mathcal{Z}[2\pi i] \subset \mathcal{Z}[2\pi i]. \tag{0.17}$$

Towards a Broadhurst–Kreimer conjecture for A-elliptic multiple zeta values

To a multiple zeta value $\zeta(k_1, \dots, k_n)$ as in (0.1), one can associate two integers, namely the weight $k_1 + \dots + k_n$ and the depth n. While the weight conjecturally defines a grading in the sense that there are no non-trivial $\mathbb{Q}$-linear relations between multiple zeta values of different weights, the depth is a rough measure for the complexity of a multiple zeta value. An important conjecture on the number of linearly independent multiple zeta values of a fixed weight and depth is due to Broadhurst and Kreimer [12], which also implies an earlier conjecture of Zagier [84] on the number of linearly independent multiple zeta values of a fixed weight.

The notions of weight and depth have analogues for elliptic multiple zeta values, namely the weight and the length. In the case of A-elliptic multiple zeta values, the analogy between length and depth is very tight, which is why we restrict to A-elliptic multiple zeta values for now.

Denote by $\mathcal{EZ}_k^{\mathrm{A}}$ the $\mathbb{Q}$-vector subspace of $\mathcal{EZ}^{\mathrm{A}}$ spanned by all A-elliptic multiple zeta values of weight k. The following conjecture is the analogue of the well-known weight grading conjecture for multiple zeta values (cf. Conjecture 1.1.1).

Conjecture. *The subspaces $\mathcal{EZ}_k^{\mathrm{A}} \subset \mathcal{EZ}^{\mathrm{A}}$ are in direct sum, i.e.*

$$\mathcal{EZ}^{\mathrm{A}} = \bigoplus_{k \geq 0} \mathcal{EZ}_k^{\mathrm{A}}. \tag{0.18}$$

Contrary to the case of multiple zeta values, the $\mathbb{Q}$-vector space $\mathcal{EZ}_k^{\mathrm{A}}$ is in general infinite-dimensional. However, the subspace $\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}})$, of A-elliptic multiple zeta values of weight k and length at most n is finite-dimensional. In analogy with the Broadhurst–Kreimer conjecture for multiple zeta values (cf. Conjecture 1.1.3), it is natural to set the following

Goal. *Find and prove a formula for*

$$D_{k,n}^{ell} := \dim_{\mathbb{Q}} \left[\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}}) / \mathcal{L}_{n-1}(\mathcal{EZ}_k^{\mathrm{A}}) \right], \tag{0.19}$$

for all $k, n \geq 0$.[2]

In this thesis, we prove a formula for $D_{k,n}^{ell}$ for $n \leq 2$ and all $k \geq 0$ and prove the weight-grading conjecture above in the special case of length at most two. The precise result is the following theorem, which has also been published by the author in [59].

[2] Here, we set $\mathcal{L}_{-1}\mathcal{EZ}_k^{\mathrm{A}} = \{0\}$ for all $k \geq 0$. We also note that $\mathcal{L}_n\mathcal{EZ}_0^{\mathrm{A}} = \mathbb{Q}$ for all n.

Theorem 4 (cf. Theorems 3.1.9, 4.6.1, and 4.2.1)**.** (i) *We have*

$$D^{\mathrm{ell}}_{k,1} = \begin{cases} 1 & \text{if } k \geq 2 \text{ is even,} \\ 0 & \text{else.} \end{cases} \tag{0.20}$$

and

$$D^{\mathrm{ell}}_{k,2} = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \left\lfloor \frac{k}{3} \right\rfloor + 1 & \text{if } k \text{ is odd.} \end{cases} \tag{0.21}$$

(ii) *The* $\mathbb{Q}$*-vector subspaces* $\mathcal{L}_2(\mathcal{EZ}^{\mathrm{A}}_k) \subset \mathcal{EZ}^{\mathrm{A}}$ *are in direct sum.*

For (i), the proof of the $\leq$-inequality proceeds in two steps. We first introduce the *Fay-shuffle space* $\mathrm{FSh}_2(d)$, a length two elliptic analogue of the double shuffle space [48], and prove that $D^{\mathrm{ell}}_{k,2} \leq \mathrm{FSh}_2(k-2)$ for all $k \geq 0$. Then, we use the representation theory of the symmetric group S_3 to find that $\dim_{\mathbb{Q}} \mathrm{FSh}_2(k-2) = \left\lfloor \frac{k}{3} \right\rfloor + 1$. This part of the proof is structurally very reminiscent of work on the Broadhurst–Kreimer conjecture [48, 49].

For the $\geq$-inequality, we use again the differential equation for elliptic multiple zeta values, in the form given in [32]. This implies that the derivative of an elliptic double zeta value is given by a $\mathbb{Q}[2\pi i]$-linear combination of Eisenstein series. Since the Eisenstein series are linearly independent over $\mathbb{C}$, it is enough to study the matrix of coefficients obtained from the derivatives of the A-elliptic double zeta values $I^{\mathrm{A}}(r, k-r)$, for $0 \leq r \leq \lfloor \frac{k}{3} \rfloor$. It turns out that the rank of this matrix is large enough, i.e. is at least $\left\lfloor \frac{k}{3} \right\rfloor + 1$, which yields the $\geq$-inequality, and thus proves the theorem. The weight-grading conjecture in length two is also proved by the differential equation, using in addition the transcendence of π.

Finally, we mention that we have also obtained a partial result in length three, cf. Section 4.7.

The meta-abelian quotient of the elliptic KZB associator and periods of Eisenstein series

The last result obtained in this thesis that we discuss in this introduction is an analogue for the elliptic KZB associator $(\Phi_{\mathrm{KZ}}, \underline{A}(\tau), \underline{B}(\tau))$ of Drinfeld's formula expressing the Drinfeld associator Φ_{KZ} in terms of the Gamma function ([30], §3, and (1.30) in this thesis).

Let

$$\underline{\mathfrak{A}}(\tau) = \log(\underline{A}(\tau)), \quad \underline{\mathfrak{B}}(\tau) = \log(\underline{B}(\tau)). \tag{0.22}$$

These are both formal Lie series, in other words, $\underline{\mathfrak{A}}(\tau), \underline{\mathfrak{B}}(\tau) \in \widehat{\mathcal{L}}$, where $\widehat{\mathcal{L}}$ denotes the graded completion of the free $\mathbb{C}$-Lie algebra on two generators x_0, x_1 (cf. Appendix A.1), which is a topological Lie algebra with (completed) Lie bracket $[\cdot,\cdot]$. We are interested in the *meta-abelian quotient* $\widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)}$ of $\widehat{\mathcal{L}}$, where $\widehat{\mathcal{L}}^{(2)} = [\widehat{\mathcal{L}}^{(1)}, \widehat{\mathcal{L}}^{(1)}]$ and $\widehat{\mathcal{L}}^{(1)}$ denotes the commutator. With hindsight towards eliminating cumbersome occurrences of $2\pi i$ from the formulae, we make a change of coordinates $a := 2\pi i x_0$ and $b = x_1$.
It is known that there is a canonical isomorphism

$$\widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)} \cong (\mathbb{C}a \oplus \mathbb{C}b) \oplus \mathbb{C}[[A, B]], \tag{0.23}$$

of $\mathbb{C}$-vector spaces (cf. Appendix E, Section 3). Thus an element $f \in \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)}$ can be identified with a pair

$$(f^{(0)}, f^{(1)}), \quad f^{(0)} \in \mathbb{C}a \oplus \mathbb{C}b,\ f^{(1)} \in \mathbb{C}[[A, B]]. \tag{0.24}$$

In particular, considering the images of $\underline{\mathfrak{A}}(\tau)$ and $\underline{\mathfrak{B}}(\tau)$ in $\widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)}$, we obtain pairs of elements

$$(\underline{\mathfrak{A}}(\tau)^{(0)}, \underline{\mathfrak{A}}(\tau)^{(1)}), \quad (\underline{\mathfrak{B}}(\tau)^{(0)}, \underline{\mathfrak{B}}(\tau)^{(1)}). \tag{0.25}$$

It is easy to see that $\underline{\mathfrak{A}}(\tau)^{(0)} = -b$ and $\underline{\mathfrak{B}}^{(0)}(\tau) = a - \tau b$. The computation of $\underline{\mathfrak{A}}(\tau)^{(1)}$ and $\underline{\mathfrak{B}}(\tau)^{(1)}$ is more elaborate, and it turns out that both can be expressed using a very particular subclass of the iterated Eisenstein integrals (0.4) and Riemann single zeta values. The precise result is as follows.

Theorem 5 (cf. Appendix E, Theorem 1.2). *Let $\underline{\mathfrak{A}}_\infty^{(1)}$ be the value of $\underline{\mathfrak{A}}(\tau)^{(1)}$ at the tangential base point $\overrightarrow{1}_\infty$ at $i\infty$ [20], and define $\underline{\mathfrak{B}}_\infty^{(1)}$ likewise. We have:*

(i)

$$\underline{\mathfrak{A}}(\tau)^{(1)} = \underline{\mathfrak{A}}_\infty^{(1)} + \sum_{m\geq 0,\, n\geq 1} \frac{2}{(m+n-1)!} \alpha_{m,n}(\tau) \left(-B\frac{\partial}{\partial A}\right)^{n-1} A^{m+n-1}B, \tag{0.26}$$

where

$$\underline{\mathfrak{A}}_\infty^{(1)} = -\left(\sum_{k\geq 2} \lambda_k A^{k-1} + \frac{1}{4}B - \sum_{k\geq 3,\mathrm{odd}} \frac{\zeta(k)}{(2\pi i)^k} B^k\right) \tag{0.27}$$

$\lambda_k := \frac{B_k}{k!}$*, and* $\alpha_{m,n}(\tau) = -\mathcal{E}(\{0\}_{n-1}, m+n+1; \tau) + \frac{B_{m+n+1}}{2(m+n+1)}\mathcal{E}(\{0\}_n; \tau)$*. In particular,* $\alpha_{m,n}(\tau) = 0$*, if* $m+n \geq 1$ *is even.*

(ii)

$$\underline{\mathfrak{B}}(\tau)^{(1)} = \underline{\mathfrak{B}}^{(1)}_\infty - \sum_{r\geq 1}\mathcal{E}(\{0\}_r;\tau)\sum_{m,n\geq 0} c_{m,n}\left[\left(-B\frac{\partial}{\partial A}\right)^r A^m B^n\right]$$
$$+\left[\sum_{k\geq 1}\frac{2}{(2k-2)!}\left\{\mathcal{E}(\{0\}_{r-1},2k;\tau)+\frac{1}{2k-1}\mathcal{E}(\{0\}_{r-2},2k,0;\tau)\right\}\right.$$
$$\left.\times\left(-B\frac{\partial}{\partial A}\right)^{r-1}A^{2k-1}\right], \tag{0.28}$$

where

$$\underline{\mathfrak{B}}^{(1)}_\infty = -\left(\sum_{k\geq 2}\lambda_k B^{k-1} + \sum_{k\geq 3,\,\mathrm{odd}}\frac{\zeta(k)}{(2\pi i)^k}AB^{k-1} + \sum_{m,n\geq 2}\lambda_m\lambda_n A^m B^{n-1}\right). \tag{0.29}$$

Here, we set $\mathcal{E}(\{0\}_{-1},2k,0;\tau) := 0$*, and* $c_{m,n}$ *is defined as the coefficient of* $A^m B^n$ *in* $\underline{\mathfrak{B}}^{(1)}_\infty$*.*

Note that the "terms at infinity" $\underline{\mathfrak{A}}^{(1)}_\infty$ and $\underline{\mathfrak{B}}^{(1)}_\infty$ together retrieve the *extended period polynomials* of Eisenstein series [82]. Written in homogeneous coordinates A, B, the extended period polynomial $r_{E_{2k}}(A,B)$ of the Eisenstein series E_{2k}, for $k \geq 2$, equals

$$r_{E_{2k}}(A,B) = \omega^+_{E_{2k}} p^+_{E_{2k}}(A,B) + \omega^-_{E_{2k}} p^-_{E_{2k}}(A,B), \tag{0.30}$$

with

$$p^+_{E_{2k}}(A,B) = A^{2k-2} - B^{2k-2} \tag{0.31}$$
$$p^-_{E_{2k}}(A,B) = \sum_{-1\leq n\leq 2k-1}\lambda_{n+1}\lambda_{2k-1-n}A^n B^{2k-2-n}, \tag{0.32}$$

where $\lambda_k := \frac{B_k}{k!}$, and the numbers $\omega^\pm_{E_{2k}} \in \mathbb{C}$ are given by $\omega^+_{E_{2k}} = \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}\omega^-_{E_{2k}}$, $\omega^-_{E_{2k}} = -\frac{(2k-2)!}{2}$. Strictly speaking, $r_{E_{2k}}$ is not a polynomial, but lives in the slightly bigger space $\bigoplus_{-1\leq n\leq 2k-1}\mathbb{C}\cdot A^n B^{2k-2-n}$, hence the name "extended period polynomial".

Theorem 6 (cf. Appendix E, Theorem 1.3)**.** *The extended period polynomial of the Eisenstein series* E_{2k} *equals*

$$\frac{(2k-2)!}{2}\left(\widetilde{\mathfrak{A}}_{2k-2}(A,B)^+ + \widetilde{\mathfrak{B}}_{2k-2}(B,A)^+ + \widetilde{\mathfrak{A}}_{2k-2}(A,B)^- + \widetilde{\mathfrak{B}}_{2k-2}(A,B)^-\right), \tag{0.33}$$

where

$$\widetilde{\mathfrak{A}}(A,B) = \frac{1}{B}\underline{\mathfrak{A}}^{(1)}_\infty(A,B), \quad \widetilde{\mathfrak{B}}(A,B) = \frac{1}{A}\underline{\mathfrak{B}}^{(1)}_\infty(A,B), \tag{0.34}$$

a subscript $2k-2$ *denotes the homogeneous component of degree* $2k-2$ *and a superscript* $+$*, resp.* $-$ *denotes the invariants, resp. anti-invariants, with respect to* $(A,B)\mapsto(-A,B)$*.*

The interpretation of period polynomials of Eisenstein series as constant terms in the elliptic KZB associator points to a relationship between elliptic multiple zeta values and the theory of universal mixed elliptic motives [45]. More precisely, one has the notion of period of a universal mixed elliptic motive, and the periods of Eisenstein series are particularly simple examples of such periods (cf. [45], §9 and [44] §11). It would be very interesting to find out the precise relationship between elliptic multiple zeta values and periods of universal mixed elliptic motives.

Conclusion and future directions

We give a summary of the results of this thesis, and indicate some possible directions for future research.

Decomposition of elliptic multiple zeta values

In this thesis, we have defined and studied two algebras of elliptic multiple zeta values, $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$, as well as their variants $\overline{\mathcal{EZ}}^{\mathrm{A}}$ an $\overline{\mathcal{EZ}}^{\mathrm{B}}$. In particular, we have exhibited explicit embeddings

$$\iota_A:\overline{\mathcal{EZ}}^{\mathrm{A}}\hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^\vee\otimes\mathcal{Z}[2\pi i]\hookrightarrow T(\mathbf{e})^\vee\otimes\mathcal{Z}[2\pi i] \tag{0.35}$$

$$\iota_B:\overline{\mathcal{EZ}}^{\mathrm{B}}\hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^\vee\otimes\mathcal{Z}\hookrightarrow T(\mathbf{e})^\vee\otimes\mathcal{Z} \tag{0.36}$$

into a space of words in formal variables $\mathbf{e}_{2k}$, $k\geq 0$, and with multiple zeta values (including $2\pi i$) as coefficients. These embeddings correspond to the representation of elliptic multiple zeta values as linear combinations of iterated Eisenstein integrals and multiple zeta values. A description of the image of these maps amounts to identify elliptic multiple zeta values among iterated Eisenstein integrals and multiple zeta values, and we have partially achieved this goal, by considering the image of ι_A (resp. the image of ι_B) in two complementary quotients of $U(\mathfrak{u}^{\mathrm{geom}})^\vee\otimes\mathcal{Z}[2\pi i]$ (resp. of $U(\mathfrak{u}^{\mathrm{geom}})^\vee\otimes\mathcal{Z}$). A complete characterization of the images of ι_A and ι_B will be the subject of a joint work with Lochak and Schneps [56].

Length-graded A-elliptic multiple zeta values

As another aspect of the study of the $\mathbb{Q}$-algebra $\mathcal{EZ}^{\mathrm{A}}$, we have also undertaken a first step to state and prove an analogue of the Broadhurst–Kreimer conjecture about the dimensions $D^{\mathrm{ell}}_{k,n}$ of the spaces of (length-graded) A-elliptic multiple zeta values. Unlike the situation for multiple zeta values, we have not been able to give a conjectural formula for $D^{\mathrm{ell}}_{k,n}$. On the other hand, in contrast to the formulas for the dimensions of depth-graded multiple zeta values predicted by the Broadhurst–Kreimer conjecture, which are only known to give upper bounds, we have actually proved a formula for $D^{\mathrm{ell}}_{k,2}$ for all k. It would be interesting to further pursue the goal of finding (and eventually proving) a Broadhurst–Kreimer type conjecture for elliptic multiple zeta values.

Multiple elliptic polylogarithms at torsion points and beyond

The elliptic multiple zeta values studied in this paper can be viewed as special values of multiple elliptic polylogarithms [23], evaluated at the point 0 of an elliptic curve. A possible venue for further research would be to study the special values of multiple elliptic polylogarithms evaluated at *torsion points* of an elliptic curve, i.e. points $\xi = s + r\tau \in E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, such that s and r are rational numbers. We tentatively call this generalization of elliptic multiple zeta values *twisted elliptic multiple zeta values*.

Extending the scope of elliptic multiple zeta values to twisted elliptic multiple zeta values is the analogue of the passage from multiple zeta values to *cyclotomic multiple zeta values*, which are the special values of multiple polylogarithms evaluated at torsion points of the punctured complex plane $\mathbb{C}^\times$, a.k.a roots of unity. Usually, one restricts to N-torsion, where N is some positive integer, and one obtains N-cyclotomic multiple zeta values, the case of $N = 1$ corresponding to the multiple zeta values. The algebraic structure of the algebra of N-cyclotomic multiple zeta values has, at least conjecturally, a similar structure as the algebra of multiple zeta values. In particular, there is an analogue of the Zagier conjecture for N-cyclotomic multiple zeta values [29]. For more on N-cyclotomic multiple zeta values, see the articles [27, 38, 39].

Finally, since elliptic multiple zeta values are in some sense a genus one analogue of multiple zeta values, a perhaps rather ambitious goal would be to find a good analogue of multiple zeta values for curves of higher genera. A closely related notion should be multiple polylogarithms for higher genus curves, a definition of which was proposed in [40].

Content

In Chapter 1, we give a brief introduction to multiple zeta values. In order to streamline the presentation, we have chosen to focus on the results and conjectures on multiple zeta values, whose analogues for elliptic multiple zeta values are studied in this thesis. Then, in Chapter 2, we set the stage for the introduction of elliptic multiple zeta values. First, we recall the definition of a classical Kronecker series [78, 82], and of a certain family of differential one-forms on a once-punctured elliptic curve [23], which are basic to the definition of elliptic multiple zeta values. We conclude the first chapter with introducing, following [31], the elliptic KZB-associator, which will play the role of the generating series of elliptic multiple zeta values.
Chapter 3, introduces elliptic multiple zeta values and the $\mathbb{Q}$-algebras generated by them. We begin with the case of A-elliptic multiple zeta values, since the algebra generated by A-elliptic multiple zeta values admits a rather simple presentation, reminiscent of the algebra of multiple zeta values. The definition of A-elliptic multiple zeta values is borrowed from Enriquez [32], however, the notion of B-elliptic multiple zeta values we use in this thesis is not exactly the one used in [32], although it is inspired by Enriquez's previous work [31]. Chapter 3 consists mainly of calculations and explicit formulae, for example we classify A-elliptic single zeta values (the length one case) by giving explicit formulae in terms of powers of $2\pi i$ (Proposition 3.1.8).
Chapter 4 contains our work on the elliptic analogue of the Broadhurst–Kreimer conjecture. Its content is essentially the paper [59], however the specific presentation is slightly different in keeping with the overall thrust of this thesis. The main result is the proof of Theorem 4.
In Chapter 5, we extend the study of Enriquez's differential equation to all lengths. This leads us naturally to study the decomposition of elliptic multiple zeta values as linear combinations of iterated Eisenstein integrals and multiple zeta values. We present partial results towards a solution of the problem of distinguishing elliptic multiple zeta values among arbitrary linear combinations of iterated Eisenstein integrals and multiple zeta values. More precisely, we prove Theorems 2 and 3. These results relate elliptic multiple zeta values to period polynomials of modular forms. Chapter 5 can be seen simultaneously as a natural extension of the article [14], and

as a precursor to a future work [56].

In Appendix A, we first collect some background on two notions, which are pervasive throughout the entire thesis, namely Lie algebras on the one hand, and iterated integrals on the other hand. None of the results presented there is original, and this section is merely intended as a useful first reference, and as a guide to the literature on these subjects.

The rest of the appendix reproduces a total of four papers by the author (two as a co-author).

Appendix B contains the paper "Linear independence of indefinite iterated Eisenstein integrals", in which we prove Theorem 1. A previous version had been uploaded to the arXiv with identifier arXiv:1601.05743.

In Appendix C, we have the paper "Elliptic multiple zeta values and one-loop open superstring amplitudes", co-authored with J. Broedel, C.R. Mafra and O. Schlotterer. This is the first account of how elliptic multiple zeta values arise in string theory. The paper has been published in the Journal of High Energy Physics, see the bibliography item [13] for details. It is also available from the arXiv, identifier arXiv:1412.5535v2, and Appendix C contains the arXiv version of the article.

Appendix D contains the paper "Relations between elliptic multiple zeta values and a special derivation algebra", co-authored with J. Broedel and O. Schlotterer. This paper has been published in Journal of Physics A, see the bibliography item [14] for details. Like Appendix C, Appendix D contains the arXiv version of the article, which is available from the arXiv, identified arXiv:1507.02254v2.

Finally, Appendix E contains the paper "The meta-abelian elliptic KZB associator and periods of Eisenstein series", where we prove Theorems 5 and Theorem 6. This paper has been uploaded to the arXiv with identifier arXiv:1608.00740.

Chapter 1

Aspects of the theory of multiple zeta values

In this chapter, we give a brief introduction to the algebra of multiple zeta values. We have streamlined the presentation, so that it connects well with our own study on elliptic multiple zeta values, which is undertaken in the main body of this thesis. For a more exhaustive introduction to multiple zeta values, we refer to Waldschmidt's lecture notes [76]; see also Deligne's Bourbaki talk [28] for a thoroughly geometric perspective on multiple zeta values.

1.1 The conjectures of Zagier and Broadhurst–Kreimer

For integers $k_1, \ldots, k_n \geq 1$ with $k_n \geq 2$, one defines the *multiple zeta value*[1]

$$\zeta(k_1, \ldots, k_n) = \sum_{0<m_1<\ldots<m_n} \frac{1}{m_1^{k_1} \ldots m_n^{k_n}}. \tag{1.1}$$

The multiple zeta value $\zeta(k_1, \ldots, k_n)$ is said to have *weight* $k_1 + \ldots + k_n$ and *depth* n. One also includes the case $n = 0$, by setting $\zeta() = 1$. One denotes by

$$\mathcal{Z} := \mathrm{Span}_{\mathbb{Q}}\{\zeta(k_1, \ldots, k_n)\} \subset \mathbb{R} \tag{1.2}$$

the $\mathbb{Q}$-vector space spanned by all multiple zeta values, which is in fact even a $\mathbb{Q}$-subalgebra of $\mathbb{R}$: the product of any two multiple zeta values can again be written as a $\mathbb{Q}$-linear combination of multiple zeta values [48].

[1] The order of summation varies in the literature. Our conventions are compatible for example with [40] and [16]

The $\mathbb{Q}$-algebra $\mathcal{Z}$ carries extra structure, corresponding to the notions of weight and depth of multiple zeta values. First, for every $k \geq 0$, define the subspace

$$\mathcal{Z}_k := \mathrm{Span}_{\mathbb{Q}}\{\zeta(k_1, \ldots, k_n) \,|\, k_1 + \ldots + k_n\} \subset \mathcal{Z} \tag{1.3}$$

of multiple zeta values of weight k. The $\mathbb{Q}$-algebra structure on $\mathcal{Z}$ is compatible with the weight in the sense that $\mathcal{Z}_k \cdot \mathcal{Z}_{k'} \subset \mathcal{Z}_{k+k'}$, for all $k, k' \geq 0$. The following conjecture is well-known, see for example [40], Conjecture 1.1a).

Conjecture 1.1.1 ("Weight-grading conjecture")**.** *The subspaces $\mathcal{Z}_k \subset \mathcal{Z}$ are in direct sum, i.e.*

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k. \tag{1.4}$$

In order to appreciate the strength of this conjecture, note that it immediately implies the transcendence of all special values $\zeta(2k+1)$ of the Riemann zeta function at positive odd integers[2]. Indeed, if $P(X)$ is a polynomial with $\mathbb{Q}$-coefficients, then the equation $P(\zeta(2k+1)) = 0$ yields a $\mathbb{Q}$-linear relation between multiple zeta values of different weights, hence $P(X) \equiv 0$ by the weight-grading conjecture. However, none of the $\zeta(2k+1)$ has been proven to be transcendental so far, the strongest result in this direction being Apéry's theorem [3] that $\zeta(3) \notin \mathbb{Q}$.
Another fundamental conjecture gives a precise formula for the dimension $\dim_{\mathbb{Q}} \mathcal{Z}_k$ of the weight k-component of $\mathcal{Z}$ [84].

Conjecture 1.1.2 (Zagier conjecture)**.** *Let $(d_k)_{k \geq 0}$ be the sequence, defined recursively by*

$$d_k = d_{k-2} + d_{k-3}, \quad k \geq 3 \tag{1.5}$$

with initial conditions $d_0 = d_2 = 1$ and $d_1 = 0$. Then

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k. \tag{1.6}$$

Equivalently, we have

$$\sum_{k \geq 0} (\dim_{\mathbb{Q}} \mathcal{Z}_k) x^k = \frac{1}{1 - x^2 - x^3}. \tag{1.7}$$

A further refinement of the Zagier conjecture involves the depth, and is due to Broadhurst and Kreimer [12, 48]. For $n \geq 0$, let

$$\mathcal{Z}_k^{(n)} = \mathrm{Span}_{\mathbb{Q}}\{\zeta(k_1, \ldots, k_r) \in \mathcal{Z}_k \,|\, r \leq n\} \tag{1.8}$$

[2] The transcendence of the even values follows from Euler's result that $\zeta(2k) \in \mathbb{Q}^\times \cdot \pi^{2k}$, along with the transcendence of π.

be the subspace of $\mathcal{Z}_k$ spanned by multiple zeta values of depth at most n (we also set $\mathcal{Z}_k^{(-1)} := \{0\}$ for all k). Then the dimension of the quotient space $D_{k,n} := \dim_{\mathbb{Q}}(\mathcal{Z}_k^{(n)}/\mathcal{Z}_k^{(n-1)})$ equals the cardinality of a basis of multiple zeta values of weight k and depth n, which cannot be expressed using multiple zeta values of weight k and depth strictly smaller than n.

Conjecture 1.1.3 (Broadhurst–Kreimer conjecture, Version 1)**.** *We have*

$$\sum_{k,n\geq 0} D_{k,n}x^k y^n = \frac{1+E(x)y}{1-O(x)y+S(x)y^2-S(x)y^4}, \tag{1.9}$$

where $E(x) = \frac{x^2}{1-x^2}$, $O(x) = \frac{x^3}{1-x^2}$ *and* $S(x) = \frac{x^{12}}{(1-x^4)(1-x^6)}$.

Upon setting $y = 1$, the Broadhurst–Kreimer conjecture retrieves the Zagier conjecture (1.7). Also, note that the series $S(x)$ occurring in the Broadhurst–Kreimer conjecture is precisely the generating series of the dimensions of the space of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. Explanations of this phenomenon are given in [20, 37, 45, 68].
The Broadhurst–Kreimer conjecture has another facet, which is related to the problem of determining the number of algebra generators of $\mathcal{Z}$. In order to formulate it, let $\mathcal{I} = \bigoplus_{k\geq 1}\mathcal{Z}_k$ be the ideal of $\mathcal{Z}$, consisting of multiple zeta values of strictly positive degree (the *augmentation ideal* of $\mathcal{Z}$). The square $\mathcal{I}^2$ is then the ideal of all (non-trivial) products in $\mathcal{Z}$. Also, define for $k, n \geq 0$ the $\mathbb{Q}$-vector space

$$\mathcal{M}_k^{(n)} = \mathcal{Z}_k^{(n)}/(\mathcal{Z}_k^{(n-1)} + \mathcal{Z}_k^{(n)} \cap \mathcal{I}^2). \tag{1.10}$$

This is the space of all multiple zeta values of depth equal to n and weight equal to k, which cannot be written as products of other multiple zeta values of depths $\leq n$. Hence, $g_{k,n} := \dim_{\mathbb{Q}} \mathcal{M}_k^{(n)}$ equals the number of algebra generators of $\mathcal{Z}$ of depth n and weight k.

Conjecture 1.1.4 (Broadhurst–Kreimer conjecture, Version 2)**.** *We have*

$$\sum_{k,n\geq 0} D_{k,n}x^k y^n = (1+E(x)y)\prod_{k\geq 3,\, n\geq 1}\frac{1}{(1-x^k y^n)^{g_{k,n}}}. \tag{1.11}$$

1.2 Towards a conceptual understanding of the conjectures

Although so far none of the conjectures described in the last section have been settled completely, there have been some advances towards a solution. Concerning the Zagier conjecture, Conjecture 1.1.2, the following theorem was proved independently by Deligne-Goncharov [29] and by Terasoma [74].

Theorem 1.2.1 (Deligne-Goncharov, Terasoma)**.** *Let d_k be the sequence defined in* (1.5)*. We have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k, \tag{1.12}$$

for all $k \geq 0$.

Both proofs use elaborate techniques from algebraic geometry, most prominently the category $MT(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ [2, 26, 29, 39]. It is known [26, 29, 39] that the category $MT(\mathbb{Z})$ is equivalent to the category of finite-dimensional representations of a pro-affine algebraic group $\mathcal{G}_{MT(\mathbb{Z})}$, whose affine ring of functions is (non-canonically) isomorphic to

$$\mathcal{H}^{MT(\mathbb{Z})} := \mathbb{Q}\langle\mathcal{F}\rangle \otimes \mathbb{Q}[f_1, f_1^{-1}], \quad \mathcal{F} = \{f_3, f_5, f_7, \ldots\}. \tag{1.13}$$

Here, $\mathbb{Q}\langle\mathcal{F}\rangle$ denotes the free shuffle algebra on the set $\mathcal{F}$ [66], and the f_i have *weight* i. The algebra $\mathcal{H}^{MT(\mathbb{Z})}$ is graded for the weight, and one can show that the subalgebra

$$\mathcal{H}^{MT(\mathbb{Z})+} := \mathbb{Q}\langle\mathcal{F}\rangle \otimes \mathbb{Q}[f_2] \subset \mathcal{H}^{MT(\mathbb{Z})}, \quad f_2 := -\frac{f_1^2}{24} \tag{1.14}$$

satisfies the Zagier conjecture, in the sense that the sequence $(\dim_{\mathbb{Q}} \mathcal{H}_k^{MT(\mathbb{Z})+})_{k\geq 0}$ of its graded components satisfies the recursion (1.5). As a consequence, the following conjecture (cf, [40], Conjecture 1.1b)) implies both the weight-grading conjecture, and the Zagier conjecture for multiple zeta values.

Conjecture 1.2.2 (Goncharov)**.** *There exists an isomorphism of $\mathbb{Q}$-algebras*

$$\phi : \mathcal{Z} \xrightarrow{\cong} \mathcal{H}^{MT(\mathbb{Z})+}, \tag{1.15}$$

which respects the weight, i.e. for every $k \geq 0$, the morphism ϕ restricts to an isomorphism

$$\phi_k : \mathcal{Z}_k \xrightarrow{\cong} \mathcal{H}_k^{MT(\mathbb{Z})+} \tag{1.16}$$

between the respective weight k-components.

Brown [16] has proved that Goncharov's conjecture holds upon replacing $\mathcal{Z}$ by the algebra of *motivic multiple zeta values* $\mathcal{Z}^{\mathfrak{m}}$ [16, 21, 39]. Intuitively, a motivic multiple zeta value $\zeta^{\mathfrak{m}}(k_1, \ldots, k_n)$ is the "Galois orbit" of the multiple zeta value $\zeta(k_1, \ldots, k_n)$ under the action of the group $\mathcal{G}_{MT(\mathbb{Z})}$. We will not go further into the slightly delicate construction, and instead refer to [16, 21] for a precise definition.

Theorem 1.2.3 (Brown). *There exists a (non-canonical) isomorphism of graded $\mathbb{Q}$-algebras*

$$\phi^{\mathfrak{m}} : \mathcal{Z}^{\mathfrak{m}} \xrightarrow{\cong} \mathcal{H}^{MT(\mathbb{Z})_+}, \tag{1.17}$$

which sends $\zeta^{\mathfrak{m}}(2n+1)$ to f_{2n+1} and $\zeta^{\mathfrak{m}}(2n)$ to $\frac{\zeta(2n)}{\zeta(2)^n} f_2^n$. Furthermore, there is an algorithm for constructing such an isomorphism $\phi^{\mathfrak{m}}$.

The first part of the above theorem is proved in [16], the algorithm for the computation of $\phi^{\mathfrak{m}}$ is described in [17]. It is stressed in *loc.cit.* that the construction of ϕ is not canonical, and depends on the choice of an algebra basis for $\mathcal{Z}^{\mathfrak{m}}$, which in turn exists by the main result of [16]. The image of a (motivic) multiple zeta values under such a map $\phi^{\mathfrak{m}}$ is also called its *f-alphabet representation* [70] (with respect to the choice of $\phi^{\mathfrak{m}}$). Granting a version of Grothendieck's period conjecture [2], Brown's theorem would imply the Conjecture 1.2.2 [28].

We now change our focus from the Zagier conjecture towards the Broadhurst–Kreimer conjecture (Conjectures 1.1.3 and 1.1.4). Much work has centered around the computation of the number $g_{k,n}$ of free algebra generators of $\mathcal{Z}$ (i.e. algebraically independent elements, which generate $\mathcal{Z}$) of weight k and depth n (cf. (1.11)). For example, we have the following quite general theorem, which generalizes Euler's result that every double zeta value of odd weight is a polynomial in single zeta values (cf. e.g. [37]).

Theorem 1.2.4 (Tsumura). *If $k \not\equiv n \mod 2$, then*

$$g_{k,n} = 0. \tag{1.18}$$

This result was previously known as the *parity conjecture* [11]. Tsumura's original proof [75] uses analytic methods; a purely algebraic proof is given in [18].
It therefore remains to compute $g_{k,n}$ in the case where k and n have the same parity modulo 2. This problem is wide open in general, however, we have the following results by Zagier for $n = 2$ [83] and by Goncharov for $n = 3$ [40].

Theorem 1.2.5. *It is*

$$g_{k,2} \le \left\lfloor \frac{k-2}{6} \right\rfloor, \quad k \text{ even} \tag{1.19}$$

$$g_{k,3} \le \left\lfloor \frac{(k-3)^2-1}{48} \right\rfloor, \quad k \text{ odd.} \tag{1.20}$$

The right hand sides of the above inequalities can be interpreted as dimensions of a certain bi-graded vector space, the *double shuffle space* [18, 48]. More precisely, the dimension of the degree $(k-n, n)$-component $DSh_n(k-n)$ of the double shuffle space gives an upper bound for $D_{k,n}$ [48]. Here, $DSh_n(d)$ is a certain subspace of the space $V_n(d)$ of homogeneous polynomials of degree d in n variables. The key to the upper bound result above is that on the one hand, the defining equations for DSh encode the *linearized double shuffle equations* between multiple zeta values [48], and on the other hand, for $n \leq 3$ and all $d \geq 0$, the numbers $\dim_{\mathbb{Q}} DSh_n(d)$ can be computed using representation theory of finite groups [49]. We note that Goncharov's original proof is rather different, and establishes a relation between multiple zeta values and the cohomology of certain modular varieties for GL_n [38]. For $n \geq 4$, no result analogous to the above theorem seems to be known.
Finally, we note in passing that DSh also carries the structure of a bi-graded Lie algebra under the Ihara bracket [50, 62], and that the Broadhurst–Kreimer conjecture can be rephrased as a statement about the homology of DSh, which sheds some more light on the conjecture [18, 33].

1.3 The Drinfeld associator

Instead of the algebra $\mathcal{Z}$ of multiple zeta values, it is frequently useful to consider the generating series of all multiple zeta values as follows. Let $\mathbb{Q}\langle x_0, x_1\rangle$ denote the $\mathbb{Q}$-vector space, freely spanned by all words w on the letters x_0, x_1 (including the empty word), which carries a natural commutative product, namely the *shuffle product* [48]. Following [48], one now defines the map

$$\zeta^{ш} : \mathbb{Q}\langle x_0, x_1\rangle \to \mathcal{Z} \tag{1.21}$$

to be the unique $\mathbb{Q}$-algebra homomorphism, which sends x_0 and x_1 to zero, and such that $\zeta^{ш}(x_0^{k_n-1}x_1 \dots x^{k_1-1}x_1) = (-1)^n\zeta(k_1, \dots, k_n)$, for $k_n \geq 2$. The formal power series

$$\Phi_{\mathrm{KZ}}(x_0, x_1) := \sum_{w \in \langle x_0, x_1\rangle} \zeta^{ш}(w) \cdot w, \tag{1.22}$$

in the non-commuting variables x_0, x_1 is called the *Drinfeld associator* [30]. It is known that the Drinfeld associator arises in various different contexts, for example in the study of representation of braid groups [30, 53] and also in Grothendieck-Teichmüller theory [35]. However, Drinfelds original definition didn't use multiple zeta values, and the interpretation of Φ_{KZ} as a generating series of multiple zeta values was only given later in [34, 53]. Instead, Drinfeld defined the Drinfeld associator as a special monodromy of the *Knizhnik-Zamolodchikov equation* [52][3]

$$\frac{\partial}{\partial z}h(z) = \left(\frac{x_0}{z} + \frac{x_1}{z-1}\right) \cdot h(z), \quad f : U \to \mathbb{C}\langle\langle x_0, x_1 \rangle\rangle, \tag{1.23}$$

where $U := \mathbb{C} \setminus \{(-\infty, 0] \cup [1, \infty)\}$. More precisely, there exist unique solutions to (1.23), which satisfy $h_0(z) \sim z^{x_0}$, as $z \to 0$ and $h_1(z) \sim (1-z)^{x_1}$, as $z \to 1$, meaning that the function $h_0(z)z^{-x_0}$ resp. $h_1(z)(1-z)^{-x_1}$ has an analytic continuation in a small neighborhood of 0 resp. 1. Then

$$\Phi_{\mathrm{KZ}} := h_1^{-1}h_0 \in \mathbb{C}\langle\langle x_0, x_1 \rangle\rangle. \tag{1.24}$$

Remark 1.3.1. Consider the differential one-form

$$\omega_{\mathrm{KZ}} = \frac{\mathrm{d}z}{z}x_0 + \frac{\mathrm{d}z}{z-1}x_1. \tag{1.25}$$

Then (1.23) can be reformulated as the equation

$$\mathrm{d}h(z) = \omega_{\mathrm{KZB}} \cdot h(z), \quad h : U \to \mathbb{C}\langle\langle x_0, x_1 \rangle\rangle, \tag{1.26}$$

where U is as above. Using the general theory of iterated integrals to solve linear differential equations, explained in Appendix A.2, one can show that

$$\Phi_{\mathrm{KZ}}^{op} = \sum_{n=0}^{\infty} \int_0^1 \omega_{\mathrm{KZ}}^n, \tag{1.27}$$

where the iterated integral has to be regularized with respect to the tangential base points 1 at 0 and -1 at 1 (cf. Appendix A.2.4), and the superscript *op* denotes the opposite multiplication on $\mathbb{C}\langle\langle x_0, x_1 \rangle\rangle$. Comparing coefficients with (1.22) gives the integral representation for multiple zeta values, first described by Kontsevich. For example, for $k_1, \ldots, k_n \geq 1$ with $k_n \geq 2$, we have

$$\zeta(k_1, \ldots, k_n) = (-1)^n \int_0^1 \omega_1 \omega_0^{k_1-1} \ldots \omega_1 \omega_0^{k_n-1}, \tag{1.28}$$

where $\omega_i = \frac{\mathrm{d}z}{z-i}$.

[3] To be precise, Knizhnik-Zamolodchikov and also Drinfeld use the variables $x_0' = x_0/(2\pi i)$ and $x_1' = x_1/(2\pi i)$ instead. This introduces some extra powers of $2\pi i$ in the denominators, which are absent here.

Many properties of the Drinfeld associator have been found and described in [30], for example there is a connection between Φ_{KZ} and the classical Gamma function [79]

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\mathrm{d}t, \tag{1.29}$$

which can be described as follows. First, one can define $\varphi_{\mathrm{KZ}} := \log(\Phi_{\mathrm{KZ}})$ by the usual formal power series for the logarithm. A priori, φ_{KZ} is only contained in $\mathbb{C}\langle\langle x_0, x_1\rangle\rangle$, however, one can show that φ_{KZ} is a *Lie-series*, i.e. $\varphi_{\mathrm{KZ}} \in \widehat{\mathcal{L}}$, where $\widehat{\mathcal{L}}$ denotes the graded completion of the free $\mathbb{C}$-Lie algebra on the generators x_0, x_1. This is a topological Lie algebra, whose (complete) Lie bracket will also be denoted by $[\cdot,\cdot]$. Letting $\widehat{\mathcal{L}}^{(1)} \subset \widehat{\mathcal{L}}$ be the commutator, Drinfeld proves that $\varphi_{\mathrm{KZ}} \in \widehat{\mathcal{L}}^{(1)}$, and goes on to compute the image of φ_{KZ} in the double-commutator quotient $\widehat{\mathcal{L}}^{(1)}/[\widehat{\mathcal{L}}^{(1)}, \widehat{\mathcal{L}}^{(1)}] \cong \mathbb{C}[[X_0, X_1]]$.

Theorem 1.3.2 (Drinfeld)**.** *Let $\varphi^{(1)}_{\mathrm{KZ}}$ denote the image of φ_{KZ} in $\mathbb{C}[[X_0, X_1]]$. We have*

$$\begin{aligned}\varphi^{(1)} &= \frac{1}{X_0X_1}\left[\exp\left(\sum_{n=2}^\infty \frac{\zeta(n)}{n}(X_0^n + X_1^n - (X_0+X_1)^n)\right) - 1\right] \\ &= \frac{1}{X_0X_1}\left[\frac{\Gamma(1-X_0)\Gamma(1-X_1)}{\Gamma(1-(X_0+X_1))} - 1\right].\end{aligned} \tag{1.30}$$

Here, one uses the formal Taylor expansion of the logarithm of the Gamma function [79]

$$\log(\Gamma(1-z)) = \gamma z + \sum_{n=2}^\infty \frac{\zeta(n)}{n} z^n, \tag{1.31}$$

which converges absolutely for $|z| < 1$.

Finally, we mention that the Drinfeld associator is only a special case of the more general notion of an associator, also introduced in [30], and that it was shown in [36] that a suitable factorization into a product of Gamma functions as in Theorem 1.3.2 holds all associators (in the sense of [30]).

Chapter 2

Towards elliptic multiple zeta values

The aim of this chapter is to put together the mathematical background needed for the study of elliptic multiple zeta values, and to describe the greater mathematical contexts in which elliptic multiple zeta values take their place. For example, elliptic multiple zeta values are on the one hand related to (multiple) elliptic polylogarithms, on which an extensive and further growing literature exists [6, 8, 9, 23, 54, 80, 81], and on the other hand to the *universal elliptic Knizhnik-Zamolodchikov-Bernard (KZB) connection* [24, 43, 45], which furnishes a natural genus one analogue of the Knizhnik-Zamolodchikov (KZ) connection, which appeared in the context of multiple zeta values (1.23). For both objects, an important ingredient is a certain Kronecker series [78, 82], represented by a quotient of Jacobi theta functions.

2.1 A Kronecker series

Fix τ in the upper half-plane $\mathbb{H} := \{\xi \in \mathbb{C} \mid \mathrm{Im}(\xi) > 0\}$. We consider a version of the odd Jacobi theta function θ_τ, defined for $\xi \in \mathbb{C}$ by

$$\theta_\tau(\xi) := \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} z^{(n+\frac{1}{2})}, \quad q = e^{2\pi i\tau},\ z = e^{2\pi i\xi}. \tag{2.1}$$

Definition 2.1.1. The *Kronecker series* $F_\tau : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ (see [78, 82][1]) is the meromorphic function

$$F_\tau(\xi,\alpha) = \frac{\theta'_\tau(0)\theta_\tau(\xi+\alpha)}{\theta_\tau(\xi)\theta_\tau(\alpha)}. \tag{2.2}$$

[1] The definition of F_τ given in [82] differs slightly from the one given here. If F_τ^{Zag} denotes the variant introduced in [82], then we have $F_\tau(\xi,\alpha) = F_\tau^{\mathrm{Zag}}(2\pi i\xi, 2\pi i\alpha)$.

The properties of the Kronecker series that we will use are summarized in the following proposition.

Proposition 2.1.2. *The Kronecker series F_τ has the following properties.*

(i) *For any $m, n \in \mathbb{Z}$, it has a simple pole at $\xi = m + n\tau$ with residue $e^{-2\pi i n\alpha}$, a simple pole at $\alpha = m + n\tau$ with residue $e^{-2\pi i\xi}$ and a simple zero at $\xi + \alpha = m + n\tau$. Moreover it has no other poles nor zeros.*

(ii) *It satisfies $F_\tau(\xi, \alpha) = F_\tau(\alpha, \xi) = -F_\tau(-\xi, -\alpha)$.*

(iii) *It is quasi-periodic in both ξ and α:*

$$F_\tau(\xi + 1, \alpha) = F_\tau(\xi, \alpha), \quad F_\tau(\xi + \tau, w) = e^{-2\pi i \frac{\mathrm{Im}(\xi)}{\mathrm{Im}(\tau)}\alpha} F_\tau(\xi, \alpha), \tag{2.3}$$

$$F_\tau(\xi, \alpha + 1) = F_\tau(\xi, \alpha), \quad F_\tau(\xi, \alpha + \tau) = e^{-2\pi i \frac{\mathrm{Im}(\alpha)}{\mathrm{Im}(\tau)}\xi} F_\tau(\xi, \alpha). \tag{2.4}$$

(iv) *Under the natural action of the full modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C} \times \mathbb{C} \times \mathbb{H}$, it transforms as*

$$F_{\frac{a\tau+b}{c\tau+d}}((c\tau + d)^{-1}\xi, (c\tau + d)^{-1}\alpha) = (c\tau + d)e^{2\pi i c\frac{\xi\alpha}{c\tau+d}} F_\tau(\xi, \alpha), \tag{2.5}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. For short,

$$(F_\tau(\xi, \alpha))|\gamma = (c\tau + d)e^{2\pi i c\frac{\xi\alpha}{c\tau+d}} F_\tau(\xi, \alpha) \tag{2.6}$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

(v) *We have*

$$\begin{aligned} F_\tau(\xi, \alpha) &= -2\pi i \left(\frac{z}{1-z} + \frac{1}{1-w} + \sum_{m,n>0} (z^m w^n - z^{-m} w^{-n}) q^{mn} \right) \\ &= \pi i[\coth(\pi i\xi) + \coth(\pi i\alpha)] - 4\pi i \sum_{n=1}^{\infty} \sum_{d|n} \sinh\left[2\pi i \left(\frac{n}{d}\xi + d\alpha\right)\right] q^n, \end{aligned} \tag{2.7}$$

where $z = e^{2\pi i\xi}$, $w = e^{2\pi i\alpha}$ and $q = e^{2\pi i\tau}$.

(vi) *It satisfies the Fay identity:*

$$\begin{aligned} F_\tau(\xi_1, \alpha_1)F_\tau(\xi_2, \alpha_2) &= F_\tau(\xi_1, \alpha_1 + \alpha_2)F_\tau(\xi_2 - \xi_1, \alpha_2) \\ &\quad + F_\tau(\xi_2, \alpha_1 + \alpha_2)F_\tau(\xi_1 - \xi_2, \alpha_1). \end{aligned} \tag{2.8}$$

Moreover, F_τ is the unique meromorphic function satisfying (i)-(iii).

Proof: Properties (i)-(iv), the second equality in (v), as well as uniqueness are proved in [82], Theorem 3. For the first equality of (v) and the Fay identity, see [23], Proposition 5. □

By property (i) of the last proposition, we can expand F_τ as a Laurent series at $\alpha = 0$

$$F_\tau(\xi, \alpha) = \sum_{k \geq 0} f^{(k)}(\xi, \tau)\alpha^{k-1}. \tag{2.9}$$

The properties listed in Proposition 2.1.2 have straightforward analogues for the functions $f^{(k)}(\xi)$, which are given in the next proposition. In the following, we will omit the τ-dependence from the notation, and simply write $f^{(k)}(\xi)$ instead of $f^{(k)}(\xi, \tau)$.

Proposition 2.1.3. *The functions $f^{(k)}(\xi)$ have the following properties.*

(i) *They are meromorphic functions with poles only at $\xi = m + n\tau$. Moreover, for $k \neq 1$, the functions $f^{(k)}(\xi)$ have no poles for $\xi \in \mathbb{Z}$.*

(ii) *They have even (resp. odd) parity when k is even (resp. odd):*

$$f^{(k)}(-\xi) = (-1)^k f^{(k)}(\xi). \tag{2.10}$$

(iii) *They are quasi-periodic in ξ:*

$$f^{(k)}(\xi + 1) = f^{(k)}(\xi), \quad f^{(k)}(\xi + \tau) = \sum_{n=0}^{k} f^{(n)}(\xi) \left(-2\pi i \frac{\mathrm{Im}(\xi)}{\mathrm{Im}(\tau)}\right)^{k-n}. \tag{2.11}$$

(iv) *Under the natural action of the full modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C} \times \mathbb{H}$, they transform as*

$$(c\tau + d)^{-k} f^{(k)}(\xi)_{|\gamma} = \sum_{n=0}^{k} f^{(n)}(\xi) \left(2\pi i c \frac{\xi}{c\tau + d}\right)^{k-n}, \tag{2.12}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and the action of γ on the left hand side is the same as in Proposition 2.1.2.

(v) *They have Fourier expansions in $q = e^{2\pi i\tau}$:*

$$f^{(0)}(\xi) = 1, \quad f^{(1)}(\xi) = \pi i \coth(\pi i\xi) - 4\pi i \sum_{m=1}^{\infty} \sinh(2\pi i m\xi) \sum_{n=1}^{\infty} q^{mn} \tag{2.13}$$

and for $k \geq 2$:

$$f^{(k)}(\xi) = \begin{cases} -2\zeta(k) - 2\dfrac{(2\pi i)^k}{(k-1)!} \displaystyle\sum_{m=1}^{\infty} \cosh(2\pi i m\xi) \sum_{n=1}^{\infty} n^{k-1} q^{mn} & \text{if } k \text{ is even} \\ 2\dfrac{(2\pi i)^k}{(k-1)!} \displaystyle\sum_{m=1}^{\infty} \sinh(2\pi i m z) \sum_{n=1}^{\infty} n^{k-1} q^{mn} & \text{if } k \text{ is odd.} \end{cases} \tag{2.14}$$

(vi) *They satisfy the Fay identity:*

$$\begin{aligned} f^{(m)}(\xi_1) f^{(n)}(\xi_2) = & -(-1)^n f^{(m+n)}(\xi_1 - \xi_2) \\ & + \sum_{r=0}^{n} \binom{m+r-1}{m-1} f^{(n-r)}(\xi_2 - \xi_1) f^{(m+r)}(\xi_1) \\ & + \sum_{r=0}^{m} \binom{n+r-1}{n-1} f^{(m-r)}(\xi_1 - \xi_2) f^{(n+r)}(\xi_2). \end{aligned} \tag{2.15}$$

Proof: All properties follow straightforwardly from the corresponding properties in Proposition 2.1.2. For (v), we used in addition the well-known expansions

$$\pi i \coth(\pi i\xi) = \frac{1}{\xi} - 2\sum_{n\geq 1} \zeta(2n) \xi^{2n-1}, \tag{2.16}$$

as well as

$$\sinh(2\pi i(m\xi + n\alpha)) = \sum_{r,s\geq 0,\, r+s \text{ odd}} (2\pi i)^{r+s} \frac{(m\xi)^r}{r!} \frac{(n\alpha)^s}{s!}. \tag{2.17}$$

□

2.2 Iterated integrals on an elliptic curve

In this section, we describe a family of iterated integrals on a once-punctured complex elliptic curve, introduced in [23].

First, recall the general definition of an iterated integral: given smooth differential one-forms $\omega_1, \ldots, \omega_n$ on a smooth k-manifold M ($k = \mathbb{R}$ or $k = \mathbb{C}$) and a piecewise smooth path $\gamma : [0,1] \to M$, one defines the iterated integral

$$\int_\gamma \omega_1 \ldots \omega_n := \int \cdots \int_{0\leq t_1 \leq \ldots \leq t_n \leq 1} f_1(t_1) \ldots f_n(t_n) \mathrm{d}t_1 \ldots \mathrm{d}t_n, \tag{2.18}$$

where $f_i(t_i)\mathrm{d}t_i := \gamma^*(\omega_i)$. Every k-linear combination of such integrals will also be called an *iterated integral*, and by convention, the empty iterated integral $\int_\gamma$ is identically equal to one. We will denote by $H^0(\mathbb{B}(M))$ the k-vector space of all homotopy invariant iterated integrals on M. By definition, these are iterated integrals, whose value along a path is invariant under homotopies of the path (cf. Definition A.2.4). For more on iterated integrals, see Appendix A.2 and the references given therein.

2.2.1 Differential forms on a once-punctured elliptic curve

Fix again $\tau \in \mathbb{H}$, and consider the once-punctured complex elliptic curve

$$E_\tau^\times := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \setminus \{0\}. \tag{2.19}$$

In [23], an explicit description of $H^0(\mathbb{B}(E_\tau^\times))$ is given in terms of the Kronecker series (2.2). Before stating the result, we need a bit more notation. Let $\xi = s + r\tau$, with $s, r \in \mathbb{R}$ be the canonical coordinate on $E_\tau^\times$, and $\nu = 2\pi i \mathrm{d}r$. Recall from Proposition 2.1.2 that the Kronecker series $F_\tau(\xi, \alpha)$ is meromorphic with simple poles at $\xi = 0$ and $\alpha = 0$, and that it transforms quasi-periodically with respect to lattice transformations $\xi \mapsto \xi + m + n\tau$. Therefore, the differential one-form

$$\Omega_\tau(\xi, \alpha) := e^{2\pi i r \alpha} F_\tau(\xi, \alpha)\mathrm{d}\xi \tag{2.20}$$

is invariant under transformations $\xi \mapsto \xi+m+n\tau$ and real analytic on $\mathbb{C}\setminus(\mathbb{Z}+\mathbb{Z}\tau)$. Using the Laurent expansion of $F_\tau(\xi, \alpha)$, which is available from (2.9), we obtain a formal expansion

$$\Omega_\tau(\xi, \alpha) = \sum_{k\geq 0} \omega^{(k)}\alpha^{k-1}, \tag{2.21}$$

where every $\omega^{(k)}$ is a real analytic differential one-form on $E_\tau^\times$. Thus $\alpha\Omega_\tau(\xi, \alpha)$ is a $\mathbb{Q}[[\alpha]]$-valued differential one-form on $E_\tau^\times$.
Moreover the Fay identity for the Kronecker series (Proposition 2.1.2 vi)) implies the quadratic relation

$$\Omega(\xi_1, \alpha_1)\wedge\Omega(\xi_2, \alpha_2) = \Omega_\tau(\xi_1, \alpha_1+\alpha_2)\wedge\Omega_\tau(\xi_2-\xi_1, \alpha_2)+\Omega_\tau(\xi_2, \alpha_1+\alpha_2)\wedge\Omega_\tau(\xi_1-\xi_2, \alpha_1). \tag{2.22}$$

and it is immediate from the product rule that

$$\mathrm{d}\Omega_\tau(\xi, \alpha) = \nu\alpha \wedge \Omega_\tau(\xi, \alpha). \tag{2.23}$$

By [23] (3.4), $\omega^{(0)} = \mathrm{d}\xi$ and the higher $\omega^{(k)}$ satisfy

$$\mathrm{d}\omega^{(k+1)} = \nu \wedge \omega^{(k)}, \quad \forall k \geq 0. \tag{2.24}$$

2.2.2 The elliptic KZB form

In this section, we introduce the *elliptic Knizhnik-Zamolodchikov-Bernard form*, or *elliptic KZB form* for short, in the way described in [23]. We let $\widehat{\mathcal{L}}$ be the graded completion of the free $\mathbb{C}$-Lie algebra on two generators x_0, x_1 (cf. Section A.1).

Definition 2.2.1. The *elliptic KZB form* ω_{KZB} is the formal $\widehat{\mathcal{L}}$-valued differential one-form on $E_\tau^\times$, defined by

$$\omega_{\mathrm{KZB}} := -\nu x_0 + \mathrm{ad}(x_0)\Omega_\tau(\xi, \mathrm{ad}(x_0))(x_1) = -\nu x_0 + \sum_{k\geq 0} \omega^{(k)} \,\mathrm{ad}^k(x_0)(x_1). \tag{2.25}$$

Remark 2.2.2. Strictly speaking, the differential form ω_{KZB} above is a real analytic trivialization, introduced by Brown and Levin in [23], of the original universal elliptic KZB form [24, 55]. The advantage of Brown-Levin's definition is that ω_{KZB} is defined as an honest differential form on the once-punctured elliptic curve $E_\tau^\times$, at the expense of losing meromorphicity, while the original KZB form [24, 55] is meromorphic, but only quasi-periodic, and hence does not descend to the once-punctured elliptic curve.

Proposition 2.2.3. *The $\widehat{\mathcal{L}}$-valued differential form ω_{KZB} has the following properties.*

(i) *It is integrable, i.e.*

$$\mathrm{d}\omega_{\mathrm{KZB}} + \omega_{\mathrm{KZB}} \wedge \omega_{\mathrm{KZB}} = 0. \tag{2.26}$$

(ii) *Under the natural action of* $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathbb{C} \times \mathbb{H}$, *it transforms as*

$$\begin{aligned}(\omega_{\mathrm{KZB}})_{|\gamma} &= -\nu_{|\gamma} x_0 + \sum_{k\geq 0} \omega^{(k)}_{|\gamma} \,\mathrm{ad}^k(x_0)(x_1)\\ &= -2\pi i(\overline{c\tau + d})\mathrm{d}r \cdot x_0 + \sum_{k\geq 0} \omega^{(k)} (c\tau + d)^{k+1} \,\mathrm{ad}^k(x_0)(x_1).\end{aligned} \tag{2.27}$$

In particular, the differential one-form $\omega^{(k)}$ is modular of weight $k+1$.

(iii) *As $\tau \to i\infty$, ω_{KZB} degenerates to the KZ form* (1.25) *in the Lie variables* $\frac{\mathrm{ad}(x_0)}{e^{2\pi i\,\mathrm{ad}(x_0)}-1}(x_1)$ *and* $\mathrm{ad}(x_0)(x_1)$*:*

$$\lim_{\tau\to i\infty} \omega_{\mathrm{KZB}} = \frac{\mathrm{d}z}{z}\frac{\mathrm{ad}(x_0)}{e^{2\pi i\,\mathrm{ad}(x_0)} - 1}(x_1) + \frac{\mathrm{d}z}{z-1}\mathrm{ad}(x_0)(x_1), \quad z = e^{2\pi i \xi}. \tag{2.28}$$

(iv) *The image of ω_{KZB} in the abelianization $\widehat{\mathcal{L}}/[\widehat{\mathcal{L}}, \widehat{\mathcal{L}}]$ is given by*

$$\omega^{\mathrm{ab}}_{\mathrm{KZB}} = -2\pi i \mathrm{d}r \cdot x_0 + \mathrm{d}\xi \cdot x_1. \tag{2.29}$$

Proof: Properties ii) and iii) follow directly from the analogous properties of the Kronecker series (Proposition 2.1.2 iv),v)), while property iv) is immediate from the expansion

$$\omega_{\mathrm{KZB}} = -\nu x_0 + \sum_{k\geq 0} \omega^{(k)} \operatorname{ad}^k(x_0)(x_1) \equiv -2\pi i \mathrm{d}r \cdot x_0 + \mathrm{d}\xi \cdot x_1 \mod [\widehat{\mathcal{L}}, \widehat{\mathcal{L}}]. \quad (2.30)$$

Finally, property i) follows from a direct calculation, using (2.23):

$$\begin{aligned} \mathrm{d}\omega_{\mathrm{KZB}} &= \mathrm{d}\operatorname{ad}(x_0)\Omega_\tau(\xi, \operatorname{ad}(x_0))(x_1) \\ &= \nu \operatorname{ad}(x_0) \wedge \operatorname{ad}(x_0)\Omega_\tau(\xi, \operatorname{ad}(x_0))(x_1) \\ &= \nu x_0 \wedge \operatorname{ad}(x_0)\Omega_\tau(\xi, \operatorname{ad}(x_0))(x_1) + \operatorname{ad}(x_0)\Omega_\tau(\xi, \operatorname{ad}(x_0))(x_1) \wedge \nu x_0 \\ &= -\omega_{\mathrm{KZB}} \wedge \omega_{\mathrm{KZB}}. \end{aligned} \quad (2.31)$$

□

2.2.3 The theorem of Brown and Levin

Consider now the formal generating series

$$T^{\mathrm{ell}} = 1 + \sum_{k\geq 1} \int \omega_{\mathrm{KZB}}^k \in \operatorname{Hom}(PE_\tau^\times, \mathbb{C})\widehat{\otimes}\langle\langle x_0, x_1\rangle\rangle, \quad (2.32)$$

where $\operatorname{Hom}(PE_\tau^\times, \mathbb{C})$ denotes the set of $\mathbb{C}$-valued functions on the path space of $E_\tau^\times$ [25]. The series T^{ell} is obtained from iteratively integrating the universal elliptic KZB form, and expanding the result into monomials in x_0, x_1. As shown for example in [46], Section 3, it follows from the integrability condition (Proposition 2.2.3 i)) that every coefficient in T^{ell} of a word $\langle x_0, x_1\rangle$ is a homotopy invariant iterated integral on $E_\tau^\times$. In [23], it is proved that the converse is true as well, more precisely.

Theorem 2.2.4 (Brown-Levin)**.** *Every homotopy invariant iterated integral on $E_\tau^\times$ can be written as a unique $\mathbb{C}$-linear combination of the coefficients of T^{ell}.*

The series T^{ell} is thus the generating series of a $\mathbb{C}$-basis of $H^0(\mathbb{B}(E_\tau^\times))$. Since every coefficient of T^{ell} is a homotopy invariant iterated integral, for any base points $\xi, \rho \in E_\tau^\times$ we have a map

$$T^{\mathrm{ell}} : \pi_1(E_\tau^\times; \xi, \rho) \to \mathbb{C}\langle\langle x_0, x_1\rangle\rangle. \quad (2.33)$$

Remark 2.2.5. Other choices of bases for $H^0(\mathbb{B}(E_\tau^\times))$ are of course possible, and there seems to be no completely natural choice of basis. On the other hand, Proposition 2.2.3 shows that the universal elliptic KZB form has some good properties, and in particular degenerates to the KZ form in the limit $\tau \to i\infty$ (with x_0 and x_1 in the original definition of the KZ form replaced by the new variables $\frac{\mathrm{ad}(x_0)}{e^{2\pi i\,\mathrm{ad}(x_0)}-1}(x_1)$ and $\mathrm{ad}(x_0)(x_1)$ respectively). Moreover, another good property of (a slight variant of) the elliptic KZB form is that it is defined over $\mathbb{Q}$ (cf. [55], §5). We refer to the introduction of [23] for a related discussion.

2.2.4 Relation to multiple elliptic polylogarithms

The theorem of Brown and Levin given in the last subsection plays an important role in the theory of *multiple elliptic polylogarithms*. We first repeat the definition found in [23].

Definition 2.2.6 (Brown-Levin)**.** For a multi-index $(k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 1}$, define the (unregularized) multiple elliptic polylogarithm by

$$E_{k_1,\dots,k_n}(\xi_1, \dots, \xi_n; \alpha_1, \dots, \alpha_n) = \sum_{m_1,\dots,m_n \in \mathbb{Z}} u_1^{m_1} \dots u_n^{m_n} I_{k_1,\dots,k_n}(q^{m_1}t_1, \dots, q^{m_n}t_n), \tag{2.34}$$

where $u_i = e^{2\pi i \alpha_i}$ for $1 \leq i \leq n$. Here

$$I_{k_1,\dots,k_n}(t_1, \dots, t_n) = \mathrm{Li}_{k_1,\dots,k_n}\left(\frac{t_1}{t_2}, \dots, \frac{t_{n-1}}{t_n}, t_n\right) \tag{2.35}$$

denotes the classical (multi-variable) multiple polylogarithm (cf. [39])

In order to put this definition into context, it is useful to recall some of the history of (multiple) elliptic polylogarithms. The main idea that elliptic polylogarithms should be weighted averages of the usual polylogarithms dates back at least to work of Bloch [9] on the *Bloch-Wigner elliptic dilogarithm*

$$D(q; x) = \sum_{n \in \mathbb{Z}} D(q^n x), \quad D(x) = \arg(1-x)\log|x| + \Im(\mathrm{Li}_2(x)), \tag{2.36}$$

where $\mathrm{Li}_2(x) := \int_0^x \log(1-t)\frac{\mathrm{d}t}{t}$ is the classical dilogarithm (cf. [85] for an introduction).

The definition (2.36) first appeared in [9] in the context of K-theory for elliptic curves. In later works, Ramakrishnan [64] and Zagier [81] extended the definition of Bloch's to define single-valued elliptic polylogarithms $D_{a,b}(q; x)$, which generalize the Bloch-Wigner elliptic dilogarithm. It was shown later by Beilinson and Levin [8] that the Bloch-Wigner-Ramakrishnan-Zagier elliptic polylogarithms can be constructed as periods of certain extensions of mixed motives, see also [54] for a purely analytic

description. The work of Beilinson and Levin was later generalized by Wildeshaus [80], from once-punctured elliptic curves to arbitrary (relative) Shimura varieties in the general context of mixed motivic sheaves. This point of view on elliptic polylogarithms has lead to important advances in arithmetic geometry, for example towards proving the Tamagawa number conjecture [47, 51]. See also [5, 6] for recent further work on other aspects of elliptic polylogarithms.
The elliptic polylogarithms described up to now can be seen as elliptic analogues of the classical polylogarithms. On the other hand, elliptic generalizations of the multiple polylogarithms $\mathrm{Li}_{k_1,\dots,k_n}(z_1,\dots,z_n)$ seem to be not as frequently studied. In fact, they have first been proposed explicitly by Levin and Racinet [55] and their study was taken up later by Brown and Levin [23].
An important result of the paper [23] is the clarification of the connection between multiple elliptic polylogarithms, homotopy invariant iterated integrals on (the configuration space of) a once-punctured elliptic curve and the elliptic KZB form. In particular, one has the following

Theorem 2.2.7 ([23], Theorem 72 & Corollary 73.)**.** *The coefficients of T^{ell} are contained in the $\mathbb{Q}$-vector space spanned by the multiple elliptic polylogarithms.*

For a precise statement, see Theorem 72 and Corollary 73 of [23].

2.3 The elliptic KZB-associator

In this section, we introduce the elliptic KZB associator of Enriquez and state some of its properties [31]. We also show how the elliptic KZB associator can be expressed in terms of homotopy invariant iterated integrals on $E_\tau^\times$.
Let $\alpha : [0,1] \to E_\tau$ be the straight line path $s \mapsto s$, and likewise let $\beta : [0,1] \to E_\tau$ be the straight line path $s \mapsto s\cdot\tau$. Since the coordinate r vanishes along α, the pull back of ω_{KZB} (2.25) along α is given by

$$\alpha^*\omega_{\mathrm{KZB}} = \mathrm{ad}(x_0)F_\tau(s,\mathrm{ad}(x_0))(x_1)\mathrm{d}s, \tag{2.37}$$

where s denotes the natural coordinate on $[0,1]$. On the other hand, a simple calculation using the definition of ω_{KZB} shows that the pull back along β^* is given by

$$\beta^*\omega_{\mathrm{KZB}} = -2\pi i x_0\mathrm{d}s + \tau\,\mathrm{ad}(x_0)e^{2\pi i s\cdot\mathrm{ad}(x_0)}F_\tau(s\cdot\tau,\mathrm{ad}(x_0))(x_1)\mathrm{d}s. \tag{2.38}$$

Recall that $\widehat{\mathcal{L}}$ denotes the free Lie algebra on the set $\{x_0, x_1\}$, completed for the degree (where x_0 and x_1 both have degree one).

Proposition 2.3.1. *There exists a unique solution* $G : (0,1) \to \exp(\widehat{\mathcal{L}})$ *to the differential equation*

$$\mathrm{d}G(s) = \Bigg[-\operatorname{ad}(x_0)F_\tau(s, \operatorname{ad}(x_0))(x_1) \Bigg] G(s)\mathrm{d}s, \tag{2.39}$$

such that $G(s) \sim (-2\pi i s)^{-[x_0,x_1]}$ *as* $s \to 0$, *where the branch of the logarithm is chosen such that* $\log(\pm\pi i/2) = \pm i$. *Likewise, there exists a unique solution* $H : (0,1) \to \exp(\widehat{\mathcal{L}})$ *to the differential equation*

$$\mathrm{d}H(s) = \Bigg[2\pi i x_0 \mathrm{d}s - \tau \operatorname{ad}(x_0) e^{2\pi i s \cdot \operatorname{ad}(x_0)} F_\tau(s\cdot\tau, \operatorname{ad}(x_0))(x_1) \Bigg] H(s)\mathrm{d}s, \tag{2.40}$$

such that $H(s) \sim (-2\pi i s)^{-[x_0,x_1]}$ *as* $s \to 0$.

Proof: It follows from Proposition A.2.2.(vi) that for all $0 < \varepsilon << 1$, the function

$$G_\varepsilon(s) = \exp\left[\int_s^\varepsilon \operatorname{ad}(x_0)F_\tau(s, \operatorname{ad}(x_0))(x_1) \right] (-2\pi i \varepsilon)^{-[x_0,x_1]} \tag{2.41}$$

solves (2.39). Moreover, by Proposition A.2.6, the limit

$$G(s) := \lim_{\varepsilon\to 0} G_\varepsilon(s) \tag{2.42}$$

exists, and as a limit of solutions to (2.39), is also a solution to (2.39) on all of $(0,1)$. To see that it has the correct asymptotic behavior, note that

$$G(s) \sim \lim_{\varepsilon\to 0} e^{-\log(s)[x_0,x_1] + \log(\varepsilon)[x_0,x_1]} e^{-\log(-2\pi i\varepsilon)[x_0,x_1]} = e^{-\log(-2\pi i s)[x_0,x_1]} \tag{2.43}$$

as $s \to 0$. A similar argument shows that the function

$$\begin{aligned} H(s) := \lim_{\varepsilon\to 0} \exp\Bigg[\int_s^\varepsilon \Big(-2\pi i x_0 + \tau \operatorname{ad}(x_0) e^{2\pi i s\cdot \operatorname{ad}(x_0)} F_\tau(s\cdot\tau, \operatorname{ad}(x_0))(x_1) \Big) \mathrm{d}s \Bigg] \\ \times (-2\pi i \varepsilon)^{-[x_0,x_1]} \end{aligned} \tag{2.44}$$

solves (2.40), and satisfies $H(s) \sim (-2\pi i s)^{-[x_0,x_1]}$ as $s \to 0$. □

Remark 2.3.2. The asymptotic condition that $G(s) \sim (-2\pi i s)^{-[x_0,x_1]}$, $H(s) \sim (-2\pi i s)^{-[x_0,x_1]}$ as $s \to 0$ can be seen as an analogue of an initial condition for an ordinary differential equation in the presence of regular singularities. Expanding

$(-2\pi is)^{-[x_0,x_1]}$ as a formal exponential introduces $\log(-2\pi i)$-terms in the formulas above. If one passes from $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ to the Tate curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ via the exponential map $\xi \mapsto e^{2\pi i\xi}$ and transports the differential equation in Proposition 2.3.1 to the Tate curve, then one would get rid of the $\log(-2\pi i)$-terms.

We extend $G(s)$ to $(0,2)\setminus\{1\}$ by analytic continuation around the point $1 \in \mathbb{C}$ in the negative direction, i.e. along a path whose image is contained in $\{a+b\tau \mid b \geq 0\}$. Likewise, we extend $H(s)$ to $(0,2)\setminus\{1\}$ along a path whose image is contained in $\{a+b\tau \mid a \geq 0\}$. Since both $\{a+b\tau \mid a \geq 0\}$ and $\{a+b\tau \mid b \geq 0\}$ are simply connected the analytic continuation does not depend on the choice of path.

Definition 2.3.3. The elliptic KZB-associator is the triple $(\Phi_{\mathrm{KZ}}, A(\tau), B(\tau))$, where Φ_{KZ} is the Drinfeld associator and $A(\tau), B(\tau) \in \mathbb{C}\langle\langle x_0, x_1\rangle\rangle$ are formal series defined by

$$A(\tau) = G(s)^{-1}G(1+s), \quad B(\tau) = H(s)^{-1}H(1+s). \tag{2.45}$$

The above definition is the original definition of the elliptic KZB associator, as given in [31], Section 6. For our purposes, however, it will be useful to modify the definition of $A(\tau)$ and $B(\tau)$ slightly by setting

$$\underline{A}(\tau) := e^{-\pi i\,\mathrm{ad}(x_0)(x_1)}A(\tau), \quad \underline{B}(\tau) := e^{\pi i\,\mathrm{ad}(x_0)(x_1)}B(\tau). \tag{2.46}$$

Remark 2.3.4. We have given the definition of the elliptic KZB associator using explicit iterated integrals, essentially following [31]. A more conceptual way of defining it, which also clarifies the difference between $A(\tau)$, $B(\tau)$ and its underscored variants $\underline{A}(\tau)$, $\underline{B}(\tau)$ can be given as follows. Consider the tangent vector $\vec{v}_0 := (-2\pi i)^{-1}\frac{\partial}{\partial\xi}$ at $0 \in E_\tau$ (which equals the tangent vector $-\frac{\partial}{\partial z}$ at 1 on the Tate curve $\mathbb{C}^\times/q^{\mathbb{Z}}$, where $q = e^{2\pi i\tau}$, $z = e^{2\pi i\xi}$). As in (2.33), we have a morphism

$$T^{ell}_{\vec{v}} : \pi_1(E_\tau^\times; \vec{v}) \to \mathbb{C}\langle\langle x_0, x_1\rangle\rangle, \tag{2.47}$$

obtained by integrating the elliptic KZB form ω_{KZB}, where $\pi_1(E_\tau^\times;\vec{v})$ is the fundamental group of $E_\tau^\times$ with respect to the "tangential base point" $\vec{v}$ at zero ([26], §15). This is a free group on two generators α, β, which correspond to the two natural homology cycles on an elliptic curve. We then have the identities

$$T^{ell}_{\vec{v}}(\alpha) = A(\tau), \quad T^{ell}_{\vec{v}}(\beta) = B(\tau). \tag{2.48}$$

On the other hand, if instead of the fundamental group $\pi_1(E_\tau^\times;\vec{v})$ one considers the fundamental torsor of paths $\pi_1(E_\tau^\times;\vec{v},-\vec{v})$ from $\vec{v}$ to $-\vec{v}$, then

$$T^{ell}_{\vec{v},-\vec{v}}(\alpha) = \underline{A}(\tau), \quad T^{ell}_{\vec{v},-\vec{v}}(\beta) = \underline{B}(\tau). \tag{2.49}$$

This definition of the elliptic KZB associator is in some sense analogous to the definition of the Drinfeld (KZ) associator Φ_{KZ} as the image of the natural straight line path $[0,1]$ under the map $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}; \overrightarrow{1}_0, -\overrightarrow{1}_1) \to \mathbb{C}\langle\langle x_0, x_1\rangle\rangle$ obtained by integrating the KZ form ω_{KZ} (cf. Example A.2.7, or [29], §5.16).

Proposition 2.3.5. *The series $\underline{A}(\tau)$ and $\underline{B}(\tau)$ satisfy the following properties.*

(i) *We have the explicit formulae*

$$\underline{A}(\tau) = \lim_{\varepsilon\to 0}(-2\pi i\varepsilon)^{\mathrm{ad}(x_0)(x_1)} \times \exp\left[\int_{\varepsilon}^{1-\varepsilon} -\mathrm{ad}(x_0)F_\tau(s,\mathrm{ad}(x_0))(x_1)\mathrm{d}s\right](-2\pi i\varepsilon)^{-\mathrm{ad}(x_0)(x_1)}, \tag{2.50}$$

$$\underline{B}(\tau) = \lim_{\varepsilon\to 0}(-2\pi i\varepsilon)^{\mathrm{ad}(x_0)(x_1)} \times \exp\left[\int_{\varepsilon}^{(1-\varepsilon)} \Big(2\pi i x_0 - \tau\,\mathrm{ad}(x_0)e^{2\pi i s\,\mathrm{ad}(x_0)}F_\tau(s\tau,\mathrm{ad}(x_0))(x_1)\Big)\,\mathrm{d}s\right] \times (-2\pi i\varepsilon)^{-\mathrm{ad}(x_0)(x_1)}. \tag{2.51}$$

(ii) *They are exponentials of Lie series $\underline{\mathfrak{A}}(\tau), \underline{\mathfrak{B}}(\tau) : \mathbb{H} \to \widehat{\mathcal{L}}$, i.e.*

$$\underline{A}(\tau) = \exp(\underline{\mathfrak{A}}(\tau)), \quad \underline{B}(\tau) = \exp(\underline{\mathfrak{B}}(\tau)). \tag{2.52}$$

These Lie series satisfy

$$\underline{\mathfrak{A}}(\tau) \equiv -x_1 \mod [\widehat{\mathcal{L}},\widehat{\mathcal{L}}] \tag{2.53}$$

$$\underline{\mathfrak{B}}(\tau) \equiv 2\pi i x_0 - \tau x_1 \mod [\widehat{\mathcal{L}},\widehat{\mathcal{L}}]. \tag{2.54}$$

Proof: (i) follows directly from the formulas for $G(t)$ and $H(t)$ given in Proposition 2.3.1, using the composition of paths formula for iterated integrals (Proposition A.2.2.(i)). For (ii), first note that since $G(t)$ and $H(t)$ (thus also $G(1+s)$ and $H(1+s)$) are solutions to initial value problems, they are group-like by Proposition A.2.3, hence so are $\underline{A}(\tau)$ and $\underline{B}(\tau)$ as products of group-like series (where we also note that $e^{\pm\pi i\,\mathrm{ad}(x_0)(x_1)}$ is also group-like). But a group-like series is the exponential of a Lie series (cf. Proposition A.1.9).
Finally, by Proposition 2.2.3, the image of ω_{KZB} in the abelianization $\widehat{\mathcal{L}}/\mathfrak{p}$, where $\mathfrak{p}$ denotes the commutator of $\widehat{\mathcal{L}}$ is given by

$$\omega^{\mathrm{ab}}_{\mathrm{KZB}} = -2\pi i \mathrm{d}r \cdot x_0 + \mathrm{d}\xi \cdot x_1. \tag{2.55}$$

Hence

$$\underline{\mathfrak{A}}(\tau) = \log(\underline{A}(\tau)) = \log(e^{-\pi i\,\mathrm{ad}(x_0)(x_1)} \cdot G^{-1}(t)G(1+t)) \equiv -\int_t^0 x_1 \mathrm{d}s + \int_{1+t}^0 x_1 \mathrm{d}s$$
$$= -x_1 \mod \mathfrak{p}, \qquad (2.56)$$

and likewise

$$\begin{aligned}\underline{\mathfrak{B}}(\tau) = \log(\underline{B}(\tau)) &= \log(e^{\pi i\,\mathrm{ad}(x_0)(x_1)} \cdot H^{-1}(t)H(1+t)) \\ &\equiv -\int_t^0 -2\pi i x_0 + \tau x_1 \mathrm{d}s + \int_{1+t}^0 -2\pi i x_0 + \tau x_1 \mathrm{d}s \qquad (2.57) \\ &= 2\pi i x_0 - \tau x_1 \mod \mathfrak{p}.\end{aligned}$$

□

We end this section by mentioning that $\underline{A}(\tau)$ and $\underline{B}(\tau)$ also satisfy a certain differential equation on the upper half-plane, which relates them to iterated integrals of Eisenstein series [20, 57] and also to a Lie algebra of special derivations on the fundamental Lie algebra of a once-punctured elliptic curve [60, 61, 75]. We postpone its discussion to Chapter 5, where it is put into its natural context.

Chapter 3

Elliptic multiple zeta values

In this chapter, we begin our study of the coefficients of Enriquez' elliptic KZB associator, which are called *elliptic multiple zeta values*. This name is justified by the fact that the elliptic KZB associator is an elliptic analogue of the Drinfeld associator (cf. Section 1.3), whose coefficients are precisely the multiple zeta values. Elliptic multiple zeta values come in two types, namely A-elliptic and B-elliptic multiple zeta values, corresponding to the two "elliptic" parts $\underline{A}(\tau)$ and $\underline{B}(\tau)$ of the elliptic KZB associator (cf. Definition 2.3.3). In this chapter, we first define A-elliptic multiple zeta values, study their properties, and give many examples. Then, we introduce our version of B-elliptic multiple zeta values, and study its relation to the variant of B-elliptic multiple zeta values defined by Enriquez. The difference between the two versions is mainly that our version of the B-elliptic multiple zeta values is given by homotopy invariant iterated integrals, while Enriquez's version is not.

3.1 Definition and first properties of A-elliptic multiple zeta values

The A-elliptic multiple zeta values to be introduced in this section have first been defined by Enriquez [32].

Recall the definition of the formal differential one-form

$$\Omega_\tau(\xi,\alpha) = e^{2\pi i r\alpha} F_\tau(\xi,\alpha), \quad \xi = s + r\tau,\ r,s\in\mathbb{R}, \tag{3.1}$$

where F_τ denotes the Kronecker series (cf. (2.20)).

Proposition 3.1.1. *For all $\lambda,\mu\in\mathbb{C}^\times$, the limit*

$$\lim_{t\to 0}(\lambda t)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_t^{1-t} \mathrm{ad}(x_0)\Omega_\tau(\xi,\mathrm{ad}(x_0))(x_1)\right](\mu t)^{-\mathrm{ad}(x_0)(x_1)} \tag{3.2}$$

exists.

Proof: If ξ is real, then $r(\xi)=0$. Therefore

$$\mathrm{ad}(x_0)\Omega_\tau(\xi,\mathrm{ad}(x_0))(x_1)=F_\tau(\xi,\mathrm{ad}(x_0))(x_1)\mathrm{d}\xi. \tag{3.3}$$

From Proposition 2.1.2, $F_\tau(\xi,\alpha)$ has a simple pole at $\xi=0$ with residue 1. Thus, the existence of the limit follows from Proposition A.2.6 □

Definition 3.1.2. For integers $k_1,\dots,k_n\geq 0$, define the *A-elliptic multiple zeta value* $I^{\mathrm{A}}(k_1,\dots,k_n;\tau)$ to be the coefficient of $\mathrm{ad}^{k_1}(x_0)(x_1)\dots\mathrm{ad}^{k_n}(x_0)(x_1)$ in

$$\lim_{t\to 0}(-2\pi it)^{\mathrm{ad}(x_0)(x_1)}\exp\left[\int_t^{1-t}\mathrm{ad}(x_0)\Omega_\tau(\xi,\mathrm{ad}(x_0))(x_1)\right](-2\pi it)^{-\mathrm{ad}(x_0)(x_1)}, \tag{3.4}$$

which is contained in $\mathbb{C}\langle\langle x_0,x_1\rangle\rangle$. The *weight* of $I^{\mathrm{A}}(k_1,\dots,k_n)$ is the sum $k_1+\ldots+k_n$, and its *length* is n.

Remark 3.1.3. If $k_1,k_n\neq 1$, then $\omega^{(k_1)}$ and $\omega^{(k_n)}$ have no poles at 0 and at 1, and $I^{\mathrm{A}}(k_1,\dots,k_n)$ is equal to the bona fide convergent iterated integral

$$I^{\mathrm{A}}(k_1,\dots,k_n;\tau)=\int_0^1\omega^{(k_1)}\dots\omega^{(k_n)}=\int_0^1 f^{(k_1)}(\xi_1)\mathrm{d}\xi_1\dots f^{(k_n)}(\xi_n)\mathrm{d}\xi_n, \tag{3.5}$$

where the functions $f^{(k)}$ have been introduced in Section 2.1.

An important property of A-elliptic multiple zeta values is their Fourier expansion.

Proposition 3.1.4. *Every A-elliptic multiple zeta value has a Fourier expansion*

$$\sum_{n=0}^{\infty}a_nq^n,\quad q=e^{2\pi i\tau}. \tag{3.6}$$

Proof: By Proposition 2.1.3, the functions $f^{(k)}(\xi)$ (which implicitly depend on τ) have Fourier expansions in $q=e^{2\pi i\tau}$. Using the equality (3.5), this Fourier expansion passes to the A-elliptic multiple zeta values by integration. □

The Fourier coefficients of A-elliptic multiple zeta values turn out to be $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of multiple zeta values, which can be computed explicitly, see Section 3.3.

Definition 3.1.5. Define the $\mathbb{Q}$-vector space of A-elliptic multiple zeta values to be

$$\mathcal{EZ}^{\mathrm{A}}=\langle I^{\mathrm{A}}(k_1,\dots,k_n;\tau)\,|\,k_1,\dots,k_n\geq 0\rangle_{\mathbb{Q}}. \tag{3.7}$$

In analogy to the case of multiple zeta values, we define for $k, n \geq 0$ the $\mathbb{Q}$-vector subspace

$$\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}}) = \langle I^{\mathrm{A}}(k_1, \ldots, k_r; \tau) \,|\, k_1 + \ldots + k_n = k,\ r \leq n \rangle_{\mathbb{Q}} \subset \mathcal{EZ}^{\mathrm{A}}. \tag{3.8}$$

We will sometimes also use the notation $\mathcal{L}_n(\mathcal{EZ}^{\mathrm{A}}) := \sum_{k \geq 0} \mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}})$ for the space of all A-elliptic multiple zeta values of length at most n.

Proposition 3.1.6. *For all $k, k', n, n' \geq 0$, we have*

$$\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}})\mathcal{L}_{n'}(\mathcal{EZ}_{k'}^{\mathrm{A}}) \subset \mathcal{L}_{n+n'}(\mathcal{EZ}_{k+k'}^{\mathrm{A}}), \tag{3.9}$$

i.e. $\mathcal{EZ}^{\mathrm{A}}$ is a bi-filtered $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathbb{H})$, the $\mathbb{C}$-algebra of holomorphic functions on $\mathbb{H}$. More precisely, we have

$$I^{\mathrm{A}}(k_1, \ldots, k_r; \tau) I^{\mathrm{A}}(k_{r+1}, \ldots, k_{r+s}; \tau) = \sum_{\sigma \in \Sigma(r,s)} I^{\mathrm{A}}(k_{\sigma(1)}, \ldots, k_{\sigma(r+s)}; \tau) \tag{3.10}$$

where $\Sigma(r, s)$ denotes the set of (r, s)-shuffles, i.e. the set of permutations σ of $\{1, \ldots, r+s\}$, such that σ^{-1} is strictly increasing on $\{1, \ldots, r\}$ and on $\{r+1, \ldots, r+s\}$.

Proof: As a quotient of Jacobi theta functions, which are holomorphic functions of τ, the Kronecker series $F_\tau(\xi, \alpha)$ is holomorphic in the variable τ as well. This implies that A-elliptic multiple zeta values are also holomorphic in τ, being integrals of holomorphic functions. Equation (3.10) follows from the definition of $I^{\mathrm{A}}(k_1, \ldots, k_n; \tau)$ and the fact that (3.4) is group-like, and hence its coefficients satisfy the shuffle product formula (by Proposition A.1.7). □

Proposition 3.1.7. *For all $k_1, \ldots, k_n \geq 0$, we have the reflection relation*

$$I^{\mathrm{A}}(k_1, \ldots, k_n; \tau) = (-1)^{k_1 + \ldots + k_n} I^{\mathrm{A}}(k_n, \ldots, k_1; \tau) \tag{3.11}$$

Proof: By the inversion of paths formula for iterated integrals (cf. Proposition A.2.2.(ii)), we have for every $0 < t << 1$

$$\int_t^{1-t} \omega^{(k_n)} \ldots \omega^{(k_1)} = (-1)^n \int_{1-t}^{t} \omega^{(k_1)} \ldots \omega^{(k_n)}. \tag{3.12}$$

It follows from the symmetry properties of the Kronecker series (Proposition 2.1.2) that under the substitution $\xi \mapsto 1 - \xi$, we have

$$\omega^{(k_i)}_{\xi \mapsto 1-\xi} = (-1)^{k_i+1} \omega^{k_i}, \tag{3.13}$$

hence

$$(-1)^n \int_{1-t}^{t} \omega^{(k_1)} \dots \omega^{(k_n)} = (-1)^{k_1+\dots+k_n} \int_{t}^{1-t} \omega^{(k_1)} \dots \omega^{(k_n)}. \tag{3.14}$$

Now $I^{\mathrm{A}}(k_n, \dots, k_1; \tau)$ is the coefficient of $\mathrm{ad}^{k_n}(x_0)(x_1) \dots \mathrm{ad}^{k_1}(x_0)(x_1)$ in

$$\lim_{t\to 0} (-2\pi i t)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_t^{1-t} \mathrm{ad}(x_0)\Omega_\tau(\xi, \mathrm{ad}(x_0))(x_1)\right](-2\pi i t)^{-\mathrm{ad}(x_0)(x_1)}. \tag{3.15}$$

and by (3.4), $(-1)^{k_1+\dots+k_n} I^{\mathrm{A}}(k_1, \dots, k_n; \tau)$ is the coefficient of

$$\left[\lim_{t\to 0} (-2\pi i t)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_t^{1-t} \mathrm{ad}(x_0)\Omega_\tau(\xi, \mathrm{ad}(x_0))(x_1)\right](-2\pi i t)^{-\mathrm{ad}(x_0)(x_1)}\right]^{op}, \tag{3.16}$$

where a supscript *op* denotes the opposite multiplication on $\mathbb{C}\langle\langle x_0, x_1\rangle\rangle$, defined by $(F \cdot G)^{op} := G \cdot F$. Comparing coefficients yields the result. □

3.1.1 Explicit examples in lengths one and two

Using the Fourier expansion of the Kronecker series, one can give explicit formulas of elliptic multiple zeta values of length one, and relate them to even single zeta values.

Proposition 3.1.8. *We have*

$$I^{\mathrm{A}}(k;\tau) = \begin{cases} -2\zeta(k) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \tag{3.17}$$

In particular, $I^{\mathrm{A}}(0;\tau) = -2\zeta(0) = -2\left(-\frac{1}{2}\right) = 1.$

Proof: For even k, we have from Proposition 2.1.3.(v)

$$\begin{aligned} I^{\mathrm{A}}(k;\tau) &= \int_0^1 \omega^{(k)} = \int_0^1 f^{(k)}(\xi)\mathrm{d}\xi \\ &= \int_0^1 \left(-2\zeta(k) - 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\cosh(2\pi i m\xi)\sum_{n=1}^{\infty} n^{k-1}q^{mn}\right)\mathrm{d}\xi \\ &= -2\zeta(k), \end{aligned} \tag{3.18}$$

while for odd k with $k \neq 1$

$$\begin{aligned} I^{\mathrm{A}}(k;\tau) &= \int_0^1 \omega^{(k)} = \int_0^1 f^{(k)}(\xi)\mathrm{d}\xi \\ &= \int_0^1 \left(-2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sinh(2\pi i m\xi)\sum_{n=1}^{\infty} n^{k-1}q^{mn}\right)\mathrm{d}\xi \end{aligned} \tag{3.19}$$

$$= 0.$$

For $k = 1$, use the Fourier expansion of $f^{(1)}$ together with (3.4) to obtain

$$\begin{aligned} I^{\mathrm{A}}(1;\tau) &= \lim_{t\to 0} \log(-2\pi i t) - \log(-2\pi i t) + \int_t^{1-t} f^{(1)}(\xi)\mathrm{d}\xi \\ &= \lim_{t\to 0} \int_t^{1-t} \pi i \coth(\pi i \xi) - 4\pi i \sum_{m=1}^{\infty} \sinh(2\pi i m \xi) \sum_{n=1}^{\infty} q^{mn} \mathrm{d}\xi \qquad (3.20) \\ &= 0. \end{aligned}$$

□

The preceding proposition already implies our first theorem concerning the algebraic structure of elliptic multiple zeta values.

Theorem 3.1.9. *We have*

$$\mathcal{L}_1(\mathcal{EZ}_k^{\mathrm{A}}) = \begin{cases} \mathbb{Q}\cdot\pi^k & \text{if } k \text{ is even} \\ \{0\} & \text{else.} \end{cases} \qquad (3.21)$$

In particular,

$$D_{k,1}^{\mathrm{ell}} = \begin{cases} 1 & \text{if } k \geq 2 \text{ is even} \\ 0 & \text{else.} \end{cases} \qquad (3.22)$$

In principle, the Fourier expansion method can be used in higher lengths as well, however, the resulting formulas become quite long and cumbersome. Later on, a different representation of A-elliptic multiple zeta values in terms of iterated integrals of Eisenstein series will be introduced.

For the case of length two elliptic multiple zeta values, we have the following partial result. See Propositions 4.1.2 and 4.1.1 for more precise results.

Proposition 3.1.10. *Let $r, s \geq 0$ with $r+s$ even. Then*

$$I^{\mathrm{A}}(r,s;\tau) = \frac{1}{2} I^{\mathrm{A}}(r;\tau) I^{\mathrm{A}}(s;\tau). \qquad (3.23)$$

In particular, if $r+s$ is even:

$$I^{\mathrm{A}}(r,s;\tau) = \begin{cases} 2\zeta(r)\zeta(s) & \text{if } r,s \text{ are both even} \\ 0 & \text{if } r,s \text{ are both odd.} \end{cases} \qquad (3.24)$$

Proof: From the shuffle product formula and the reflection relation, we get

$$I^{\mathrm{A}}(r;\tau)I^{\mathrm{A}}(s;\tau) = I^{\mathrm{A}}(r,s;\tau) + I^{\mathrm{A}}(s,r;\tau) = I^{\mathrm{A}}(r,s;\tau)(1+(-1)^{r+s}) = 2I^{\mathrm{A}}(r,s;\tau). \qquad (3.25)$$

Combining this with the explicit formula for $I^{\mathrm{A}}(k;\tau)$ (Proposition 3.1.8), we get the result. □

3.2 Comparison with the elliptic KZB-associator

In this section, we will relate A-elliptic multiple zeta values to coefficients of the elliptic KZB associator. The first step is undertaken in the following

Proposition 3.2.1. *We have*

$$\underline{A}(\tau) = \sum_{n\geq 0}(-1)^n \sum_{k_1,\dots,k_n} I^{\mathrm{A}}(k_1,\dots,k_n;\tau)\,\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_1}(x_0)(x_1). \tag{3.26}$$

Proof: By Proposition 2.3.5 i), we know that

$$\begin{aligned} A(\tau) = \Bigg[& e^{\pi i\,\mathrm{ad}(x_0)(x_1)} \\ & \times \lim_{\varepsilon\to 0}(-2\pi i\varepsilon)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_{\varepsilon}^{1-\varepsilon} -\mathrm{ad}(x_0)F_\tau(t,\mathrm{ad}(x_0))(x_1)\right] \\ & \times (-2\pi i\varepsilon)^{-\mathrm{ad}(x_0)(x_1)}\Bigg]^{op}. \end{aligned} \tag{3.27}$$

Thus, the proposition follows from the definition of A-elliptic multiple zeta values (3.4). □

Next, we show that the algebra spanned by the coefficients of $\underline{A}(\tau)$ equals the algebra of A-elliptic multiple zeta values, in a way compatible with the length and the weight. This is the analogue for A-elliptic multiple zeta values of the fact that the multiple zeta values are precisely the coefficients of the Drinfeld associator.

Theorem 3.2.2. *For all $k,n\geq 0$, we have equalities of vector spaces*

$$\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}}) = \langle \underline{A}(\tau)_w \,|\, w \text{ of length } n \text{ and weight } k\rangle_{\mathbb{Q}}. \tag{3.28}$$

Proof: By Proposition 3.2.1, we have

$$\underline{A}(\tau) = \sum_{k_1,\dots,k_n\geq 0} (-1)^n I^{\mathrm{A}}(k_1,\dots,k_n;\tau)\,\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_1}(x_0)(x_1). \tag{3.29}$$

If we expand all $\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_1}(x_0)(x_1)$ on the right hand side into word in x_0,x_1, then we see that for a word w of weight k and length n, every $\underline{A}(\tau)_w$ is a linear combination of I^{A}'s of the same weight and length. Thus, the inclusion "$\supset$" of the Proposition follows.

For the reverse inclusion, we show by induction on the set of multi-indices of length n that $I^{\mathrm{A}}(k_1,\dots,k_n;\tau)\in(\mathcal{EZ}^{\mathrm{A}})_k^n$. If $w = x_1^n$ (induction start), then $\underline{A}(\tau)_w =$

$(-1)^n I^{\mathrm{A}}(\underline{0};\tau)$. For the induction step, let $\underline{k} = (k_1,\dots,k_n)$ be a multi-index of length n, and assume the statement for all multi-indices $\underline{k'} < \underline{k}$ in the lexicographic ordering (induction hypothesis). Set $w = x_1 x_0^{k_1}\dots x_1 x_0^{k_n}$. Then the only products of Lie monomials, which can have w in their expansion into words are the ones which correspond to multi-indices $\underline{k'}$, which are strictly smaller in the lexicographic ordering than $\underline{k}$. Thus

$$\underline{A}(\tau)_w = (-1)^n I^{\mathrm{A}}(\underline{k};\tau) + \sum_{\underline{k'}<\underline{k}} (-1)^n \lambda_{\underline{k'}} I^{\mathrm{A}}(\underline{k'};\tau), \tag{3.30}$$

and we conclude using the induction hypothesis. □

3.3 Computing A-elliptic multiple zeta values

In this section, we describe a framework for describing A-elliptic multiple zeta values of some length n in terms of A-elliptic multiple zeta values of length $n-1$ and Eisenstein series. More precisely, it was proved by Enriquez [32] that A-elliptic multiple zeta values satisfy a (linear, first-order) differential equation, which is recursive for the length. The constant term, which is lost by differentiation, can be restored by considering the limit $\lim_{\tau\to i\infty}\underline{A}(\tau)$. This enables a "length-by-length"-study of A-elliptic multiple zeta values, the first steps of which have been carried out in [14, 59]. The results presented in this section should be compared to the slightly more general results of Chapter 5.

3.3.1 Differential equation

Let

$$\mathcal{I}^{\mathrm{A}}(X_1,\dots,X_n;\tau) = \sum_{k_1,\dots,k_n\geq 0} I^{\mathrm{A}}(k_1,\dots,k_n;\tau) X_1^{k_1-1}\dots X_n^{k_n-1} \tag{3.31}$$

be the generating series of A-elliptic multiple zeta values of length n.

Theorem 3.3.1 (Enriquez)**.** *For all $n \geq 0$, we have*

$$\begin{aligned}\frac{2\pi i\partial}{\partial\tau}\mathcal{I}^{\mathrm{A}}(X_1,\dots,X_n;\tau) &= \wp^*_\tau(X_1)\mathcal{I}^{\mathrm{A}}(X_2,\dots,X_n;\tau) - \wp^*_\tau(X_n)\mathcal{I}^{\mathrm{A}}(X_1,\dots,X_{n-1};\tau)\\ &+ \sum_{i=2}^{n}(\wp^*_\tau(X_i)-\wp^*_\tau(X_{i-1}))\mathcal{I}^{\mathrm{A}}(X_1,\dots,X_{i-1}+X_i,\dots,X_n;\tau),\end{aligned} \tag{3.32}$$

where $\wp^*_\tau(\alpha) = \sum_{k=-1}^{\infty}(2k+1)G_{2k+2}(\tau)\alpha^{2k}$ *is the modified Weierstrass* $\wp$*-function and* $G_k(\tau)$ *denotes the Eisenstein series*[1] *of weight* $k \geq 0$*:*

$$G_k(\tau) = \begin{cases} -1 & k = 0 \\ \displaystyle\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^k} & k \geq 1. \end{cases} \tag{3.33}$$

Proof: See [32], Théorème 3.2. □

Note that $G_k(\tau)$ vanishes, if k is odd. Also for $k \geq 1$, we have the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n, \tag{3.34}$$

where $\sigma_l(n) = \sum_{d|n} d^l$ denotes the l-th divisor function.
Comparing coefficients on both sides of (3.32), we obtain the following explicit formula for the τ-derivative of an individual A-elliptic multiple zeta value (cf. [14], eq. (2.47))

$$\begin{aligned} &2\pi i\frac{\partial}{\partial\tau}I^{\mathrm{A}}(k_1,\dots,k_n;\tau) \\ &\quad = k_1G_{k_1+1}(\tau)I^{\mathrm{A}}(k_2,\dots,k_n;\tau) - k_nG_{k_n+1}(\tau)I^{\mathrm{A}}(k_1,\dots,k_{n-1};\tau) \\ &\quad + \sum_{i=2}^{n}\Big\{(-1)^{k_i}(k_{i-1}+k_i)G_{k_{i-1}+k_i+1}(\tau)I^{\mathrm{A}}(k_1,\dots,k_{i-2},0,k_{i+1},\dots,k_n;\tau) \\ &\quad - \sum_{k=0}^{k_{i-1}+1}(k_{i-1}-k)\binom{k_i+k-1}{k}G_{k_{i-1}-k+1}(\tau)I^{\mathrm{A}}(k_1,\dots,k_{i-2},k+k_i,k_{i+1},\dots,k_n;\tau) \\ &\quad + \sum_{k=0}^{k_i+1}(k_i-k)\binom{k_{i-1}+k-1}{k}G_{k_i-k+1}(\tau)I^{\mathrm{A}}(k_1,\dots,k_{i-2},k+k_{i-1},k_{i+1},\dots,k_n;\tau)\Big\}\,. \end{aligned} \tag{3.35}$$

Solving (3.35) iteratively gives a recursive formula for A-elliptic multiple zeta values in terms of iterated integrals of Eisenstein series, which will be studied in detail in length two in Chapter 4 and in all lengths in Chapter 5. What is missing is the constant of integration, or more precisely, the constant term a_0 in the Fourier expansion of A-elliptic multiple zeta values (Proposition 3.1.4).

[1] The two cases $k = 1, 2$ require the Eisenstein summation prescription

$$\sum_{m,n\in\mathbb{Z}} a_{m,n} \equiv \lim_{N\to\infty}\lim_{M\to\infty}\sum_{n=-N}^{N}\sum_{m=-M}^{M} a_{m,n}.$$

3.3.2 Constant term procedure

In this section, we show how to retrieve the constant term a_0 in the Fourier expansion of A-elliptic multiple zeta values using the degeneration properties of the elliptic KZB-associator. The main result is the following proposition, which is implicitly already contained in [32].

Proposition 3.3.2. *The constant term in the Fourier expansion of* $I^{\mathrm{A}}(k_1,\dots,k_n;\tau)$ *equals* $(-1)^n$ *times the coefficient of* $\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_1}(x_0)(x_1)$ *in the series*

$$e^{\pi i t}\Phi(\tilde{y},t)e^{2\pi i\tilde{y}}\Phi(\tilde{y},t)^{-1}, \tag{3.36}$$

where $t=-[x_0,x_1]$ *and* $\tilde{y}=-\frac{\mathrm{ad}(x_0)}{e^{2\pi i\,\mathrm{ad}(x_0)}-1}(x_1)=-\frac{1}{2\pi i}\sum_{n\geq 0}\frac{B_n}{n!}(2\pi i)^n\,\mathrm{ad}^n(x_0)(x_1)$*, where* B_n *denotes the* n*-th Bernoulli number. Moreover, all other Fourier coefficients are contained in* $\mathcal{Z}[(2\pi i)^{-1}]$*.*

Proof: By Proposition 3.2.1, we know that $I^{\mathrm{A}}(k_1,\dots,k_n;\tau)$ is equal to the coefficient of $(-1)^n\,\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_n}(x_0)(x_1)$ in $\underline{A}(\tau)$. Therefore, $I_0^{\mathrm{A}}(k_1,\dots,k_n)$ can be retrieved as the coefficient of $(-1)^n\,\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_n}(x_0)(x_1)$ in the series

$$\underline{A}_\infty:=\lim_{\tau\to i\infty}\underline{A}(\tau)e^{\pi i t}\Phi(\tilde{y},t)e^{2\pi i\tilde{y}}\Phi(\tilde{y},t)^{-1}, \tag{3.37}$$

where $\tilde{y}=-\frac{\mathrm{ad}(x_0)}{e^{2\pi i\,\mathrm{ad}(x_0)}-1}(x_1)$ and $t=-\,\mathrm{ad}(x_0)(x_1)$. Since the series $\tilde{y}$ has coefficients in $\mathbb{Q}[(2\pi i)^{\pm}]$, this shows that the constant terms of A-elliptic multiple zeta values are contained in $\mathcal{Z}[(2\pi i)^{-1}]$. The fact that all higher Fourier coefficients are also contained in $\mathcal{Z}[(2\pi i)^{-1}]$ follows from this, together with the recursive differential equation (3.3.1) and the Fourier expansion of the Eisenstein series

$$G_{2k}(\tau)=2\zeta(2k)+\frac{2(2\pi i)^{2k}}{2(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}, \tag{3.38}$$

valid for all $k\geq 1$, which shows in particular that the Fourier coefficients of $G_{2k}(\tau)$ are contained in $\mathbb{Q}[2\pi i]$. □

Example 3.3.3. If $(k_1,\dots,k_n)\in(\mathbb{Z}_{\geq 0})^n$ is a multi-index with $k_i\neq 1$ for all i, then

$$I_0^{\mathrm{A}}(k_1,\dots,k_n)=\frac{1}{n!}\prod_{i=1}^{n}\frac{B_{k_i}}{k_i!}(2\pi i)^{k_i}. \tag{3.39}$$

This follows from Proposition 3.3.2, more precisely from (3.36), since if $k_i\neq 1$ for all i, then only the middle $e^{2\pi i\tilde{y}}$-factor in (3.36) contributes. Indeed, every word occurring in the series $e^{\pi i t}$, $\Phi(\tilde{y},t)$ and $\Phi(\tilde{y},t)^{-1}$ contains at least one $\mathrm{ad}(x_0)(x_1)$-factor, which corresponds to $k_i=1$ for some i.

For a general multi-index $(k_1,\ldots,k_n)$ with some entries $k_i = 1$, a closed formula seems very cumbersome to write down. Some explicit examples can be obtained by expanding (3.36) in low lengths, using a computer algebra software, like in our case Mathematica[2]

$$I_0^{\mathrm{A}}(1,0) = -\frac{i\pi}{2}\,, \qquad I_0^{\mathrm{A}}(1,0,0) = -\frac{i\pi}{4}\,, \qquad I_0^{\mathrm{A}}(1,0,0,0) = -\frac{i\pi}{12} - \frac{\zeta(3)}{24\zeta(2)}, \tag{3.40}$$

which generalizes to

$$I_0^{\mathrm{A}}(\underbrace{1,0,\ldots,0}_{r}) = -\frac{2\pi i}{4(r-1)!} + \sum_{k=1}^{\lfloor r/2\rfloor -1} \frac{1}{(r-(2k+1))!}\frac{\zeta(2k+1)}{(2\pi i)^{2k}}, \tag{3.41}$$

and shows that every odd Riemann zeta value arises as the constant term of some linear combination of A-elliptic multiple zeta value. Examples involving multiple zeta values of higher depth as opposed to Riemann single zeta values are a bit harder to come by, since one has to go to very high lengths. The first example of a multiple zeta value, which is not given by a polynomial in Riemann single zeta values occurs in weight 8, and is (for example) given by $\zeta(3,5)$ (cf. [17], Section 2.5). An A-elliptic multiple zeta value having $\zeta(3,5)$ occurring in its constant term is given by $I^{\mathrm{A}}(1,0,0,1,0,0,0,0,0;\tau)$, which has length 9. More precisely,

$$\begin{aligned} I_0^{\mathrm{A}}(1,0,0,1,0,0,0,0,0) = \frac{1}{(2\pi i)^6}\bigg(&-\zeta(3,5) - 2\zeta(2)\zeta(3)^2 - 6\pi i\zeta(3)\zeta(4) + 12\pi i\zeta(2)\zeta(5) \\ &+ \zeta(3)\zeta(5) - \frac{21}{2}\pi i\zeta(7) + 10\zeta(8)\bigg). \end{aligned} \tag{3.42}$$

In fact, we will see in Theorem 5.4.2 that, up to powers of $2\pi i$, all multiple zeta values arise as constant terms of A-elliptic multiple zeta values (a similar result holds for B-elliptic multiple zeta values).

3.3.3 Example of the Fourier expansion

In this section, we will show how to use the differential equation and constant term procedure to compute the Fourier expansion of $I^{\mathrm{A}}(0,1,0,0;\tau)$.
First, we get from (3.35) that

$$2\pi i\frac{\mathrm{d}}{\mathrm{d}\tau}I^{\mathrm{A}}(0,1,0,0;\tau) = I^{\mathrm{A}}(0,2,0;\tau) - I^{\mathrm{A}}(0,0,2;\tau). \tag{3.43}$$

[2]Mathematica is a trademark of Wolfram Research Inc. The implementation used to generate the examples was obtained in joint work with Johannes Broedel, and can be found on the DVD attached to the dissertation. In order to simplify the resulting multiple zeta values, we used the multiple zeta value data mine [10].

Differentiate the last equation once more to obtain, again from (3.35),

$$(2\pi i)^2\frac{\mathrm{d}^2}{\mathrm{d}^2\tau}I^{\mathrm{A}}(0,1,0,0;\tau)=4I^{\mathrm{A}}(0,3;\tau)-2I^{\mathrm{A}}(3,0;\tau)=6I^{\mathrm{A}}(0,3;\tau), \tag{3.44}$$

where the last equation follows from the reflection relation (Proposition 3.1.7). One last differentiation gives

$$(2\pi i)^3\frac{\mathrm{d}^3}{\mathrm{d}^3\tau}I^{\mathrm{A}}(0,1,0,0;\tau)=-18G_4(\tau)I^{\mathrm{A}}(0;\tau)-18I^{\mathrm{A}}(4;\tau)=-6(2\pi i)^4\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{3.45}$$

where in the last equation, we have used that $I^{\mathrm{A}}(2k;\tau)=-2\zeta(2k)$ (cf. Proposition 3.1.8) and the Fourier expansion of the Eisenstein series $G_4(\tau)$ (cf. (3.34)). Integrating (3.45) three times and supplementing the constant term, which by the constant term procedure is given by

$$I_0^{\mathrm{A}}(0,1,0,0)=-3\frac{\zeta(3)}{(2\pi i)^2}, \tag{3.46}$$

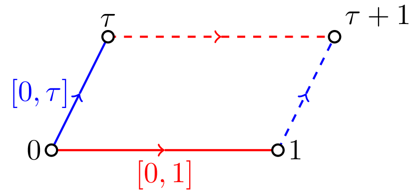
we finally obtain the desired Fourier expansion

$$I^{\mathrm{A}}(0,1,0,0;\tau)=-3\frac{\zeta(3)}{(2\pi i)^2}+\frac{6}{(2\pi i)^2}\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^n, \tag{3.47}$$

where we used that $\int_{\tau}^{i\infty}e^{2\pi i n\tau_1}\mathrm{d}\tau_1=-\frac{1}{2\pi i n}e^{2\pi i n\tau}$.

3.4 B-elliptic multiple zeta values

We have seen that A-elliptic multiple zeta values are defined as iterated integrals along the path α on a once-punctured elliptic curve $E_\tau^\times$, corresponding to the interval $[0,1]$ in $\mathbb{C}$. There is another canonical path β on $E_\tau^\times$, given by the straight line path from 0 to τ in $\mathbb{C}$. It corresponds to the series $\underline{B}(\tau)$ of the elliptic KZB associator.



Moreover, we have seen that the $\mathbb{Q}$-vector space spanned by the A-elliptic multiple zeta values equals the $\mathbb{Q}$-vector space spanned by the coefficients of the series $\underline{A}(\tau)$ arising as part of the elliptic KZB associator (cf. Definition 2.3.3).

In this section, we define B-elliptic multiple zeta values in two ways: as coefficients of $\underline{B}(\tau)$, and as iterated integrals $\int_\beta\omega^{(k_1)}\dots\omega^{(k_n)}$ of the forms $\omega^{(k)}$ along the path β. The latter definition is essentially the same as Enriquez's [32], while the first seems to be new. At the end of this section, we compare the two versions of B-elliptic multiple zeta values. Although they are rather similar, we show that they do not span the same $\mathbb{Q}$-vector space.

3.4.1 Enriquez' B-elliptic multiple zeta values

Definition 3.4.1. Define *Enriquez' B-elliptic multiple zeta value* $I^{\mathrm{B}}(k_1,\ldots,k_n;\tau)$ to be the coefficient of $\mathrm{ad}^{k_1}(x_0)(x_1)\ldots\mathrm{ad}^{k_n}(x_0)(x_1)$ in the series

$$\lim_{\varepsilon\to 0}(-2\pi i\varepsilon\tau)^{\mathrm{ad}(x_0)(x_1)}\exp\left[\int_{\varepsilon\tau}^{(1-\varepsilon)\tau}\mathrm{ad}(x_0)\Omega_\tau(\xi,\mathrm{ad}(x_0))(x_1)\right](-2\pi i\varepsilon\tau)^{-\mathrm{ad}(x_0)(x_1)}. \quad (3.48)$$

The Enriquez B-elliptic multiple zeta value $I^{\mathrm{B}}(k_1,\ldots,k_n;\tau)$ is similar to the A-elliptic multiple zeta value $I^{\mathrm{A}}(k_1,\ldots,k_n;\tau)$, the only difference being that the path $\alpha=[0,1]$ has been replaced by the path $\beta=[0,\tau]$. As for I^{A}, the *weight* of $I^{\mathrm{B}}(k_1,\ldots,k_n;\tau)$ is the sum $k_1+\ldots+k_n$, and its *length* is n. If $k_1,k_n\neq 1$, then $\omega^{(k_1)}$ and $\omega^{(k_n)}$ have no poles at 0 and 1, and $I^{\mathrm{A}}(k_1,\ldots,k_n)$ is equal to the bona fide convergent iterated integral

$$I^{\mathrm{B}}(k_1,\ldots,k_n;\tau)=\int_0^\tau\omega^{(k_1)}\ldots\omega^{(k_n)}, \quad (3.49)$$

where the path of integration is the straight line path from 0 to τ.

Definition 3.4.2. Define the $\mathbb{Q}$-vector space of Enriquez B-elliptic multiple zeta values to be

$$\mathcal{EZ}^{\mathrm{B-Enr}}=\langle I^{\mathrm{B}}(k_1,\ldots,k_n;\tau)\,|\,k_1,\ldots,k_n\geq 0\rangle_{\mathbb{Q}}. \quad (3.50)$$

and for $k,n\geq 0$, define

$$\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{B-Enr}})=\langle I^{\mathrm{B}}(k_1,\ldots,k_r;\tau)\,|\,k_1+\ldots+k_n=k,\ r\leq n\rangle_{\mathbb{Q}}\subset\mathcal{EZ}^{\mathrm{B-Enr}}. \quad (3.51)$$

Essentially the same argument as in the proof of Proposition 3.1.6 shows the

Proposition 3.4.3. *For all $k,k',n,n'\geq 0$, we have*

$$\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{B-Enr}})\mathcal{L}_{n'}(\mathcal{EZ}_{k'}^{\mathrm{B-Enr}})\subset\mathcal{L}_{n+n'}(\mathcal{EZ}_{k+k'}^{\mathrm{B-Enr}}),, \quad (3.52)$$

i.e. $\mathcal{EZ}^{\mathrm{B-Enr}}$ is a bi-filtered $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathbb{H})$, the $\mathbb{C}$-algebra of holomorphic functions on $\mathbb{H}$. More precisely, we have

$$I^{\mathrm{B}}(k_1,\ldots,k_r;\tau)I^{\mathrm{B}}(k_{r+1},\ldots,k_{r+s};\tau)=\sum_{\sigma\in\Sigma(r,s)}I^{\mathrm{B}}(k_{\sigma(1)},\ldots,k_{\sigma(r+s)};\tau), \quad (3.53)$$

where $\Sigma(r,s)$ denotes the set of (r,s)-shuffles as in Proposition 3.1.6

The next proposition gives a very precise comparison between A-elliptic and Enriquez' B-elliptic multiple zeta values. This fact already been proved by Enriquez ([32], Section 2.5), but we repeat his argument here for completeness, and also since our notation is slightly different.

Proposition 3.4.4. *For all $k_1, \dots, k_n$, we have the equality*

$$I^{\mathrm{B}}(k_1, \dots, k_n; \tau) = \tau^{k_1+\dots+k_n-n} I^{\mathrm{A}}(k_1, \dots, k_n; -1/\tau). \tag{3.54}$$

Proof: First, it follows from the modularity properties of the Kronecker series (Proposition 2.1.2 iv)) that

$$\Omega_\tau(\tau \cdot \xi, \alpha) = \tau^{-1}\Omega_{-1/\tau}(\xi, \tau^{-1} \cdot \alpha). \tag{3.55}$$

Consequently, we have the equality of generating series

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-2\pi i\varepsilon\tau)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_{\varepsilon\tau}^{(1-\varepsilon)\tau} \mathrm{ad}(x_0)\Omega_\tau(\xi, \mathrm{ad}(x_0))(x_1)\right](-2\pi i\varepsilon\tau)^{-\mathrm{ad}(x_0)(x_1)} \\
&= \lim_{\varepsilon\to 0}(-2\pi i\varepsilon)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_{\varepsilon}^{(1-\varepsilon)} \mathrm{ad}(x_0)\Omega_\tau(\tau\cdot\xi, \mathrm{ad}(x_0))(x_1)\right](-2\pi i\varepsilon)^{-\mathrm{ad}(x_0)(x_1)} \\
&= \lim_{\varepsilon\to 0}(-2\pi i\varepsilon)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_{\varepsilon}^{(1-\varepsilon)} (\tau)^{-1}\mathrm{ad}(x_0)\Omega_{-1/\tau}(\xi, (\tau)^{-1}\mathrm{ad}(x_0))(x_1)\right] \\
&\times (-2\pi i\varepsilon)^{-\mathrm{ad}(x_0)(x_1)}.
\end{aligned} \tag{3.56}$$

Now the coefficient of $\mathrm{ad}^{k_1}(x_0)(x_1)\dots\mathrm{ad}^{k_n}(x_0)(x_1)$ is precisely equal to

$$\tau^{n-(k_1+\dots+k_n)} I^{\mathrm{A}}(k_1, \dots, k_n; -1/\tau), \tag{3.57}$$

and the Proposition follows. □

Remark 3.4.5. Note that the preceding proposition does not imply that one can identify I^{A} and $I^{\mathrm{B-Enr}}$ in a straightforward fashion. Namely, the map of sets

$$\begin{aligned}
\mathcal{EZ}^{\mathrm{A}} &\to \mathcal{EZ}^{\mathrm{B-Enr}} \\
I^{\mathrm{A}}(k_1, \dots, k_n; \tau) &\mapsto I^{\mathrm{B}}(k_1, \dots, k_n; \tau)
\end{aligned} \tag{3.58}$$

is not a morphism of $\mathbb{Q}$-vector spaces, the problem being that A-elliptic multiple zeta values are not graded for the length. For example, we have

$$\frac{1}{2}I^{\mathrm{A}}(0;\tau) = \frac{1}{2} = I^{\mathrm{A}}(0,0;\tau), \tag{3.59}$$

while on the other hand

$$I^{\mathrm{B}}(0;\tau) = \tau, \quad I^{\mathrm{B}}(0,0;\tau) = \frac{\tau^2}{2}, \tag{3.60}$$

hence there is no $\mathbb{Q}$-linear relation between $I^{\mathrm{A}}(0;\tau)$ and $I^{\mathrm{B}}(0,0;\tau)$.

3.4.2 B-elliptic multiple zeta values and elliptic associators

We now propose a slightly different definition of B-elliptic multiple zeta values. Recall the definition of the series $\underline{B}(\tau)$, which was a part of the elliptic KZB associator.

Definition 3.4.6. Define the $\mathbb{Q}$-vector space of *B-elliptic multiple zeta values* to be

$$\mathcal{E}\mathcal{Z}^{\mathrm{B}} = \langle \underline{B}(\tau)_{|w} \mid w \in \langle x_0, x_1 \rangle \rangle_{\mathbb{Q}} \subset \mathcal{O}(\mathbb{H}), \tag{3.61}$$

where $\mathcal{O}(\mathbb{H})$ denotes the $\mathbb{C}$-algebra of holomorphic functions on $\mathbb{H}$.
For integers $k, n \geq 0$, we also define

$$\mathcal{L}_n(\mathcal{E}\mathcal{Z}^{\mathrm{B}}_k) := \langle B(\tau)_{|w} \mid \deg_{x_0}(w) = k,\, \deg_{x_1}(w) \leq n \rangle_{\mathbb{Q}}. \tag{3.62}$$

Since $\underline{B}(\tau)$ is a group-like series, the next proposition follows from Proposition A.1.7 and the definition of the shuffle product of words (A.16). The same method would give an alternative proof of Proposition 3.1.6.

Proposition 3.4.7. $\mathcal{E}\mathcal{Z}^{\mathrm{B}}$ *is a bi-filtered* $\mathbb{Q}$*-subalgebra of* $\mathcal{O}(\mathbb{H})$*. More precisely, for all numbers* $k, n, k', n' \geq 0$*, we have*

$$\mathcal{L}_n(\mathcal{E}\mathcal{Z}^{\mathrm{B}}_k) \cdot \mathcal{L}_{n'}(\mathcal{E}\mathcal{Z}^{\mathrm{B}}_{k'}) \subset \mathcal{L}_{n+n'}(\mathcal{E}\mathcal{Z}^{\mathrm{B}}_{k+k'}). \tag{3.63}$$

Let

$$\underline{B}^{\mathrm{Enr}}(\tau) = \sum_{n\geq 0}(-1)^n \sum_{k_1,\dots,k_n} I^{\mathrm{B}}(k_1,\dots,k_n;\tau)\,\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_1}(x_0)(x_1). \tag{3.64}$$

Comparing with Definition 3.4.1, we see that

$$\underline{B}^{\mathrm{Enr}}(\tau) = \Bigg[\lim_{\varepsilon\to 0}(-2\pi i\varepsilon\tau)^{\mathrm{ad}(x_0)(x_1)} \exp\left[\int_{\varepsilon\tau}^{(1-\varepsilon)\tau} - \mathrm{ad}(x_0)\Omega_\tau(\xi,\mathrm{ad}(x_0))(x_1)\right] \times (-2\pi i\varepsilon\tau)^{-\mathrm{ad}(x_0)(x_1)}\Bigg]^{\mathrm{op}}. \tag{3.65}$$

The following proposition has essentially already been proved by Enriquez in [32], Proposition 2.8. The proof given there can be adapted straightforwardly to our situation.

Proposition 3.4.8. *We have*

$$\underline{B}^{\mathrm{Enr}}(\tau) = \tau^{\mathrm{ad}(x_0)(x_1)} \exp\left(\frac{2\pi i}{\tau}e_+\right)\underline{B}(\tau)\tau^{-\mathrm{ad}(x_0)(x_1)}, \tag{3.66}$$

where e_+ *is the unique derivation of* $\mathcal{L}$*, which sends* $x_0 \mapsto 0$ *and* $x_1 \mapsto x_0$

Proof: By definition

$$\underline{B}(\tau) = H_1(z)^{-1}H_0(z), \tag{3.67}$$

where H_0, H_1 are solutions to the differential equation

$$\mathrm{d}H(s) = \left(2\pi i x_0 \mathrm{d}s - \tau\, \mathrm{ad}(x_0) e^{2\pi i s \cdot \mathrm{ad}(x_0)} F_\tau(s\cdot\tau, \mathrm{ad}(x_0))(x_1)\right) H(s)\mathrm{d}s \tag{3.68}$$

with asymptotics

$$\begin{aligned} H_0(z) &\sim (-2\pi i z)^{-\,\mathrm{ad}(x_0)(x_1)},\ z\to 0\\ H_1(z) &\sim (-2\pi i(1-z))^{-\,\mathrm{ad}(x_0)(x_1)},\ z\to 1. \end{aligned} \tag{3.69}$$

Applying the automorphism $\exp\left(\frac{2\pi i}{\tau}e_+\right)$ to the right hand side of (3.68), we obtain the equation

$$\mathrm{d}H^{\mathrm{Enr}}(s) = \left(-\tau\, \mathrm{ad}(x_0) e^{2\pi i s\cdot \mathrm{ad}(x_0)} F_\tau(s\cdot \tau, \mathrm{ad}(x_0))(x_1)\right) H^{\mathrm{Enr}}(s)\mathrm{d}s \tag{3.70}$$

It follows from the general theory of iterated integrals as solutions to differential equations (cf. Section A.2.2) that

$$\underline{B}^{\mathrm{Enr}}(\tau) = (H_1^{\mathrm{Enr}}(z))^{-1} H_0^{\mathrm{Enr}}(z), \tag{3.71}$$

where $H_0^{\mathrm{Enr}}(z), H_1^{\mathrm{Enr}}(z)$ are the unique solutions to (3.70) with asymptotics

$$\begin{aligned} H_0^{\mathrm{Enr}}(z) &\sim (-2\pi i\tau z)^{-\,\mathrm{ad}(x_0)(x_1)},\ z\to 0\\ H_1^{\mathrm{Enr}}(z) &\sim (-2\pi i\tau(1-z))^{-\,\mathrm{ad}(x_0)(x_1)},\ z\to 1. \end{aligned} \tag{3.72}$$

Hence

$$\exp\left(\frac{2\pi i}{\tau}\right)\underline{B}(\tau) = \tau^{-\,\mathrm{ad}(x_0)(x_1)}\underline{B}^{\mathrm{Enr}}(\tau)\tau^{\mathrm{ad}(x_0)(x_1)}, \tag{3.73}$$

where the extra factors $\tau^{\pm\,\mathrm{ad}(x_0)(x_1)}$ take into account the difference of the asymptotics between H_i^{Enr} and H_i, for $i = 0, 1$. The proposition now follows. □

3.4.3 Comparison of the two versions of B-elliptic multiple zeta values

In the last sections, we have seen two versions of B-elliptic multiple zeta values: Enriquez's version [32] as iterated integrals

$$I^{\mathrm{B}}(k_1,\ldots,k_n;\tau) = \int_\beta \omega^{(k_1)}\ldots\omega^{(k_n)} \tag{3.74}$$

on the one hand, and the coefficients of the series $\underline{B}(\tau)$ on the other. Both versions are given by values of iterated integrals along the path β on the punctured elliptic curve

$E_\tau^\times$, however the coefficients of $\underline{B}(\tau)$ are given by values of homotopy invariant iterated integrals, while the $I^{\mathrm{B}}(k_1,\dots,k_n;\tau)$ are not.
More concisely, the series $\underline{B}(\tau)$ is obtained by iterated integration of the *integrable* differential form ω_{KZB} (cf. Proposition 2.2.3), the generating series of $I^{\mathrm{B}}(k_1,\dots,k_n;\tau)$ is obtained by iterated integration of the differential form $\mathrm{ad}(x_0)\Omega_\tau(\xi,\mathrm{ad}(x_0))(x_1)\mathrm{d}\xi$, which is easily seen not to be integrable (its differential is non-zero, since it is non-holomorphic, but its wedge product with itself vanishes), and one needs to add to it the differential form $\nu = 2\pi i dr$ to obtain an integrable one-form. Therefore, it seems more reasonable for us to define B-elliptic multiple zeta values as the coefficients of $\underline{B}(\tau)$.
One may wonder whether the issue of integrability vs. non-integrability makes a difference between the two algebras $\mathcal{EZ}^{\mathrm{B}}$ and $\mathcal{EZ}^{\mathrm{B-Enr}}$. In fact, we have seen in Theorem 3.2.2 that the $\mathbb{Q}$-vector space spanned by the A-elliptic multiple zeta values $I^{\mathrm{A}}(k_1,\dots,k_n;\tau)$ equals the $\mathbb{Q}$-vector space spanned by the coefficients of $\underline{A}(\tau)$, a non-commutative power series in the variables x_0, x_1, which is part of the elliptic KZB associator. More precisely, the result is that

$$\mathcal{L}_n(\mathcal{EZ}^{\mathrm{A}}_k) = \langle \underline{A}(\tau)_w \,|\, w \text{ of length } n \text{ and weight } k\rangle_{\mathbb{Q}}, \tag{3.75}$$

where the weight of a word $w \in \langle x_0, x_1\rangle$ is its degree in x_0 and the length is its degree in x_1. But for this result to hold, we used crucially that A-elliptic multiple zeta values are special values of iterated integrals along the path α, and since the differential form ν vanishes along α, the pullbacks of ω_{KZB} and $\mathrm{ad}(x_0)\Omega_\tau(\xi,\mathrm{ad}(x_0)))(x_1)\mathrm{d}\xi$ along α agree, and consequently, the generating series of A-elliptic multiple zeta values can be identified with $\underline{A}(\tau)$ via the formula

$$\underline{A}(\tau) = \sum_{n\geq 0}(-1)^n \sum_{k_1,\dots,k_n} I^{\mathrm{A}}(k_1,\dots,k_n;\tau)\,\mathrm{ad}^{k_n}(x_0)(x_1)\dots\mathrm{ad}^{k_1}(x_0)(x_1), \tag{3.76}$$

cf. Proposition 3.2.1.
For the two versions of B-elliptic multiple zeta values, the equality 3.75 breaks down, and it is no longer true that for all $n,k\geq 0$, the space $\mathcal{L}_n(\mathcal{EZ}^{\mathrm{B-Enr}})_k$ equals

$$\mathcal{L}_n(\mathcal{EZ}^{\mathrm{B}})_k = \langle \underline{B}(\tau)_w \,|\, w \text{ of length } \leq n \text{ and weight } k\rangle_{\mathbb{Q}}. \tag{3.77}$$

For example, $\mathcal{L}_1(\mathcal{EZ}^{\mathrm{B-Enr}}_1) = \mathbb{Q}\cdot I^{\mathrm{B}}(1;\tau) = \{0\}$, by Proposition 3.4.4, using that $I^{\mathrm{A}}(1;\tau) = 0$ (cf. Proposition 3.1.8). But the word x_0 has length zero and weight one, and $\underline{B}(\tau)_{|x_0} = 2\pi i \neq 0$, by Proposition 2.3.5 ii), showing that

$$\mathcal{L}_1(\mathcal{EZ}^{\mathrm{B-Enr}})_1 \neq \langle \underline{B}(\tau)_w \,|\, w \text{ of length } \leq 1 \text{ and weight } 1\rangle_{\mathbb{Q}}. \tag{3.78}$$

Chapter 4

A-elliptic double zeta values

In this section, we study the $\mathbb{Q}$-vector space $\mathrm{gr}_2^{\mathcal{L}}(\mathcal{EZ}_k^{\mathrm{A}}) := \mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}})/\mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}})$ of *A-elliptic double zeta values* of weight $k \geq 0$. The main result is a formula for the dimension of $\mathrm{gr}_2^{\mathcal{L}}(\mathcal{EZ}_k^{\mathrm{A}})$, as well as a complete description of all $\mathbb{Q}$-linear relations between A-elliptic double zeta values.

4.1 Differential equation and constant term in length two

We begin by considering both the differential equation and the constant term procedure, given in the last section for A-elliptic multiple zeta values of arbitrary length, in full detail in the case of length two.

Proposition 4.1.1. *We have*

$$\begin{aligned} 2\pi i \frac{\partial}{\partial \tau} I^{\mathrm{A}}(0, r; \tau) &= (-1)^r 2\pi i \frac{\partial}{\partial \tau} I^{\mathrm{A}}(r, 0; \tau) \\ &= -rG_{r+1}(\tau) I^{\mathrm{A}}(0) + rG_0(\tau) I^{\mathrm{A}}(r+1), \end{aligned} \tag{4.1}$$

and if $r, s \neq 0$

$$\begin{aligned} 2\pi i \frac{\partial}{\partial \tau} I^{\mathrm{A}}(r, s; \tau) &= -sG_{s+1}(\tau) I^{\mathrm{A}}(r) + rG_{r+1}(\tau) I^{\mathrm{A}}(s) \\ &\quad - (-1)^r (r+s) G_{r+s+1}(\tau) I^{\mathrm{A}}(0) \\ &\quad + \sum_{n=1}^{r+s+1} (r+s-n) \left(\binom{n-1}{r-1} - \binom{n-1}{s-1} \right) G_{r+s+1-n}(\tau) I^{\mathrm{A}}(n), \end{aligned} \tag{4.2}$$

where $G_k(\tau)$ *denotes the Eisenstein series of weight* k *as in* (3.33)*. In particular, we have* $\frac{\partial}{\partial \tau} I^{\mathrm{A}}(r, s; \tau) = 0$ *if the weight* $r+s$ *is even.*

Proof: The equations (4.1) and (4.2) are obtained from (3.35) in the case $n = 2$. The vanishing of $\frac{\partial}{\partial \tau} I^{\mathrm{A}}(r, s; \tau)$ for $r+s$ even follows from this, bearing in mind that $G_k(\tau)$ and $I^{\mathrm{A}}(k; \tau)$ vanish, if k is odd (the latter follows from Proposition 3.1.8). □

Proposition 4.1.2. *We have*

$$I_0^{\mathrm{A}}(r,s) = \begin{cases} 0, & \text{if } r=s=1 \\ -\dfrac{B_r B_1 (2\pi i)^{r+1}}{2r!}, & \text{if } s=1 \text{ and } r \neq 1 \\ \dfrac{B_r B_s (2\pi i)^{r+s}}{2r!s!}, & \text{else}, \end{cases} \tag{4.3}$$

where the B_n are the Bernoulli numbers. In particular, we have $I^{\mathrm{A}}(1,1;\tau)=0$ and

$$I^{\mathrm{A}}(r,s;\tau) = \frac{B_r B_s (2\pi i)^{r+s}}{2r!s!}, \quad r+s \in 2\mathbb{Z},\ r\neq 1 \text{ or } s \neq 1. \tag{4.4}$$

Proof: From Proposition 3.3.2, we know that the constant term $I_0^{\mathrm{A}}(r,s)$ of $I^{\mathrm{A}}(r,s;\tau)$ is equal to the coefficient of $\mathrm{ad}^s(x_0)(x_1)\,\mathrm{ad}^r(x_0)(x_1)$ in the series

$$e^{\pi i t}\Phi(\tilde{y},t)e^{2\pi i \tilde{y}}\Phi(\tilde{y},t)^{-1}, \tag{4.5}$$

which gives (4.3). The formula for $I^{\mathrm{A}}(r,s;\tau)$ in the case $r+s$ is even is obtained from (4.3), together with Proposition 4.1.1 □

Now consider the indefinite Eisenstein integrals

$$\mathcal{G}(k;\tau) := \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \displaystyle\int_\tau^{i\infty} G_k(\tau') - 2\zeta(k)\mathrm{d}\tau' - \int_0^\tau 2\zeta(k)\mathrm{d}\tau' & \text{if } k \text{ is even.} \end{cases} \tag{4.6}$$

Integrating the Fourier expansion of the Eisenstein series $G_{2k}(\tau)$, one gets the formula[1]

$$\mathcal{G}(2k;\tau) = -2\zeta(2k)\cdot\tau + \frac{(2\pi i)^{2k-1}}{2(2k-1)!}\sum_{n=1}^{\infty}\frac{\sigma_{2k-1}(n)}{n}q^n. \tag{4.7}$$

It is easy to see that $\frac{\partial}{\partial\tau}\mathcal{G}(k;\tau) = -G_k(\tau)$ for all $k \geq 0$. Therefore, combining Propositions 4.1.1 and 4.1.2, we obtain closed formulas for A-elliptic double zeta values as linear combinations of products $I^{\mathrm{A}}(k;\tau)$ and $\mathcal{G}(k;\tau)$. Slightly more precisely, we have the following

Proposition 4.1.3. *Every A-elliptic double zeta value $I^{\mathrm{A}}(r,s;\tau)$ can be written as*

$$I_0^{\mathrm{A}}(r,s;\tau) + \frac{1}{2\pi i}\sum_{j=0}^{r+s+1}\lambda_j I^{\mathrm{A}}(j;\tau)\mathcal{G}(r+s+1-j;\tau), \tag{4.8}$$

where $I_0^{\mathrm{A}}(r,s;\tau) \in \mathbb{Q}(2\pi i)^{r+s}$. Note that $I^{\mathrm{A}}(j;\tau) \in \mathbb{Q}(2\pi i)^j$, by Proposition 3.1.8.

Also, higher length A-elliptic multiple zeta values can be written in terms of so-called *iterated Eisenstein integrals*, cf. Chapter 5.

[1]Note the exponent $2k-1$ in the prefactor $\frac{(2\pi i)^{2k-1}}{2(2k-1)!}$. This is not a typo, and comes from the fact that $\frac{\partial}{\partial\tau} = 2\pi i q\frac{\partial}{\partial q}$.

4.2 The weight-grading for A-elliptic double zeta values

Using the representation of A-elliptic double zeta values as integrals of Eisenstein series, we can prove that every $\mathbb{Q}$-linear relation between A-elliptic double zeta values decomposes into a family of $\mathbb{Q}$-linear relations, one for each weight k. The precise version is as follows.

Theorem 4.2.1. *Let $\bigoplus_{k\geq 0}\mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}})$ be the direct sum of the $\mathbb{Q}$-vector spaces $\mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}})$. The natural map*

$$\bigoplus_{k\geq 0}\mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}}) \to \sum_{k\geq 0}\mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}}) \subset \mathcal{EZ}^{\mathrm{A}} \tag{4.9}$$

induced by the inclusions $\mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}}) \subset \mathcal{EZ}^{\mathrm{A}}$ is injective.

Proof: Let

$$\sum_{i=1}^{n}\lambda_i I^{\mathrm{A}}(r_i, s_i; \tau) = 0, \quad \lambda_i \in \mathbb{Q} \tag{4.10}$$

be a $\mathbb{Q}$-linear relation between A-elliptic double zeta values. By Proposition 4.1.3, the relation (4.10) must hold simultaneously for both the constant terms of the $I^{\mathrm{A}}(r_i, s_i)$, i.e.

$$\sum_{i=1}^{n}\lambda_i I_0^{\mathrm{A}}(r_i, s_i) = 0, \tag{4.11}$$

and for the non-constant terms, which are given by Eisenstein integrals. Using that $I_0^{\mathrm{A}}(r,s) \in \mathbb{Q}(2\pi i)^{r+s}$ and the transcendence of π, we see that

$$\sum_{i=1}^{n}\lambda_i I_0^{\mathrm{A}}(r_i, s_i; \tau) = 0 \quad \Longrightarrow \quad \sum_{i,\, r_i+s_i=k}\lambda_i I_0^{\mathrm{A}}(r_i, s_i) = 0, \quad \forall k. \tag{4.12}$$

On the other hand, again by Proposition 4.1.3, the non-constant term of $I^{\mathrm{A}}(r,s;\tau)$ is given by a $\mathbb{Q}$-linear combination

$$\sum_{j=0}^{r+s+1}\mu_j(2\pi i)^j\mathcal{G}(r+s+1-j;\tau), \quad \mu_j \in \mathbb{Q} \tag{4.13}$$

of homogeneous degree $r+s+1$, where $2\pi i$ has degree one and $\mathcal{G}(k;\tau)$ has degree k. Since the family of Eisenstein series $G_{2k}(\tau)$ is linearly independent over $\mathbb{C}$, the same is true for the family of indefinite Eisenstein integrals. Moreover, it follows that the family of all products $(2\pi i)^j\mathcal{G}(2k;\tau)$ is linearly independent over $\mathbb{Q}$, again using the transcendence of π. Thus

$$\sum_{i=1}^{n}\lambda_i I^{\mathrm{A}}(r_i, s_i; \tau) = 0 \Rightarrow \sum_{i,\, r_i+s_i=k}\lambda_i I^{\mathrm{A}}(r_i, s_i; \tau) = 0, \quad \forall k, \tag{4.14}$$

which proves the result. □

4.3 Relations between elliptic double zeta values

We have already seen that A-elliptic multiple zeta values satisfy the shuffle product (Proposition 3.1.6)

$$I^{\mathrm{A}}(k_1,\ldots,k_r;\tau)I^{\mathrm{A}}(k_{r+1},\ldots,k_{r+s};\tau)=\sum_{\sigma\in\Sigma(r,s)}I^{\mathrm{A}}(k_{\sigma(1)},\ldots,k_{\sigma(r+s)};\tau) \tag{4.15}$$

as well as the reflection relation (Proposition 3.1.7)

$$I^{\mathrm{A}}(k_1,\ldots,k_n;\tau)=(-1)^{k_1+\ldots+k_n}I^{\mathrm{A}}(k_n,\ldots,k_1;\tau). \tag{4.16}$$

There exists a third type of algebraic relation between A-elliptic multiple zeta values, the *Fay relations*, which arise from the Fay identity for the Kronecker series (Proposition 2.1.2). We only treat the case of length two, and refer to [14] for Fay relations for A-elliptic multiple zeta values of higher lengths.

Proposition 4.3.1. *(Fay relation) For $r,s\geq 0$, we have*

$$\begin{aligned} I^{\mathrm{A}}(r,s;\tau)&=\delta_{r,1}\delta_{s,1}3\zeta(2)-(-1)^sI^{\mathrm{A}}(0,r+s;\tau)\\ &\quad+\sum_{n=0}^{s}(-1)^{s-n}\binom{r-1+n}{r-1}I^{\mathrm{A}}(r+n,s-n;\tau)\\ &\quad+\sum_{n=0}^{r}(-1)^{s+n}\binom{s-1+n}{s-1}I^{\mathrm{A}}(s+n,r-n;\tau). \end{aligned} \tag{4.17}$$

Proof: If $r=s=1$, then (4.17) becomes

$$I^{\mathrm{A}}(1,1;\tau)=-2I^{\mathrm{A}}(1,1;\tau)+I^{\mathrm{A}}(0,2;\tau)+2I^{\mathrm{A}}(2,0;\tau)+3\zeta(2), \tag{4.18}$$

which holds by Proposition 3.1.10. If $r\neq s$, then we can assume that $s\neq 1$, using the reflection relation $I^{\mathrm{A}}(r,s;\tau)=(-1)^{r+s}I^{\mathrm{A}}(s,r;\tau)$, if necessary. Now choose $\varepsilon>0$ and consider the function

$$\Xi_\varepsilon^{r,s}(x)=\int_\varepsilon^x f^{(s)}(\xi_2-x)\int_\varepsilon^{\xi_2}f^{(r)}(\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{4.19}$$

for $x\in[\varepsilon,1]$. Since $s\neq 1$, $f^{(s)}$ is smooth on $[0,1]$ (Proposition 2.1.3.(i)), and thus $\Xi_\varepsilon^{r,s}$ is smooth on $[0,1]$ as well. Moreover, we have $\lim_{\varepsilon\to 0}\Xi_\varepsilon^{r,s}(1)=I^{\mathrm{A}}(r,s;\tau)$, due to the periodicity of $f^{(s)}$ (Proposition 2.1.2, iii)). Now

$$\Xi_\varepsilon^{r,s}(t)=\int_\varepsilon^t(\Xi_\varepsilon^{r,s})'(x)\mathrm{d}x=\int_\varepsilon^t\int_\varepsilon^x f^{(s)}(\xi_2-x)f^{(r)}(\xi_2)\mathrm{d}\xi_2\mathrm{d}x. \tag{4.20}$$

Using the Fay identity (Proposition 2.1.2, vi)), we get

$$\Xi_\varepsilon^{r,s}(t)=\int_\varepsilon^t\int_\varepsilon^x\Bigg\{-(-1)^sf^{(r+s)}(x)+\sum_{n=0}^{s}\binom{r-1+n}{r-1}f^{(s-n)}(-x)f^{(r+n)}(\xi_2)$$

$$+\sum_{n=0}^{r}\binom{s-1+n}{s-1}f^{(s+n)}(\xi_2-x)f^{(r-n)}(x)\Bigg\}\,\mathrm{d}\xi_2\mathrm{d}x. \tag{4.21}$$

Now we evaluate both sides at $t=1$ and pass to the limit $\varepsilon\to 0$ to obtain the result. □

The reflection, shuffle and Fay relations can also be expressed as functional identities for the generating series

$$\mathcal{I}^{\mathrm{A}}(X,Y;\tau)=\sum_{k,l\geq 0} I^{\mathrm{A}}(k,l;\tau)X^{k-1}Y^{l-1}. \tag{4.22}$$

We summarize these in the following

Corollary 4.3.2. *The series $\mathcal{I}^{\mathrm{A}}(X,Y;\tau)$ satisfies the following identities.*

(i) *(Reflection relation)*

$$\mathcal{I}^{\mathrm{A}}(X,Y;\tau)=\mathcal{I}^{\mathrm{A}}(-Y,-X;\tau) \tag{4.23}$$

(ii) *(Shuffle relation)*

$$\mathcal{I}^{\mathrm{A}}(X,Y;\tau)+\mathcal{I}^{\mathrm{A}}(Y,X;\tau)=\mathcal{I}^{\mathrm{A}}(X;\tau)\mathcal{I}^{\mathrm{A}}(Y;\tau) \tag{4.24}$$

(iii) *(Fay relation)*

$$\mathcal{I}_*^{\mathrm{A}}(X,Y;\tau)+\mathcal{I}_*^{\mathrm{A}}(X+Y,-Y;\tau)+\mathcal{I}_*^{\mathrm{A}}(-X-Y,X;\tau)=0, \tag{4.25}$$

where $\mathcal{I}_*^{\mathrm{A}}(X,Y;\tau):=\mathcal{I}^{\mathrm{A}}(X,Y;\tau)+\frac{1}{2}I^{\mathrm{A}}(2;\tau)$, using that $\zeta(2)=-\frac{1}{2}I^{\mathrm{A}}(2;\tau)$ (Proposition 3.1.8).

4.4 The Fay-shuffle space

We introduce a graded $\mathbb{Q}$-vector space of rational functions, which encodes the $\mathbb{Q}$-linear relations satisfied by A-elliptic double zeta values. Denote by $\widehat{V}_d\subset\mathbb{Q}(X,Y)$ the subspace of rational functions P of degree d, such that $XY\cdot P$ is a polynomial (necessarily of degree $d+2$)

$$\widehat{V}_d=\{P\in\mathbb{Q}(X,Y)_d\,|\,XY\cdot f\in\mathbb{Q}[X,Y]\}. \tag{4.26}$$

Definition 4.4.1. For $d\geq -2$, we define the length two *Fay-shuffle space* of degree d, $\mathrm{FSh}_2(d)$, as the set of rational functions $P\in\widehat{V}_d$, which satisfy

$$P(X,Y)+P(X+Y,-Y)+P(-X-Y,X)=0,\quad P(X,Y)+P(Y,X)=0. \tag{4.27}$$

By substituting $X\mapsto -Y$ and $Y\mapsto -X$, one sees that $P\in\mathrm{FSh}_2(d)$ necessarily satisfies the reflection relation $P(X,Y)=P(-Y,-X)$. This implies the

Proposition 4.4.2. *If d is even, then* $\mathrm{FSh}_2(d) = \{0\}$.

Proof: Every $P \in \mathrm{FSh}_2(d)$ satisfies

$$P(X,Y) = P(-Y,-X), \quad P(X,Y) = -P(Y,X) \tag{4.28}$$

hence $P \equiv 0$, if d is even. □

Proposition 4.4.3. *For all $d \geq 0$, we have*

$$\dim_{\mathbb{Q}} \mathrm{gr}_2^{\mathcal{L}} \mathcal{EZ}_k^{\mathrm{A}} \leq \dim_{\mathbb{Q}} \mathrm{FSh}_2(k-2). \tag{4.29}$$

Proof: Let $(\widehat{V}_{k-2})^*$ be the dual space of $\widehat{V}_{k-2}$. Recall from the last section that the generating series $\mathcal{I}^{\mathrm{A}}(X,Y;\tau)$ of A-elliptic double zeta values satisfies the equations

$$\mathcal{I}^{\mathrm{A}}(X,Y;\tau) + \mathcal{I}^{\mathrm{A}}(Y,X;\tau) = \mathcal{I}^{\mathrm{A}}(X;\tau)\mathcal{I}^{\mathrm{A}}(Y;\tau), \tag{4.30}$$

$$\mathcal{I}_*^{\mathrm{A}}(X,Y;\tau) + \mathcal{I}_*^{\mathrm{A}}(X+Y,-Y;\tau) + \mathcal{I}_*^{\mathrm{A}}(-X-Y,X;\tau) = 0, \tag{4.31}$$

where $\mathcal{I}_*^{\mathrm{A}}(X,Y;\tau) := \mathcal{I}^{\mathrm{A}}(X,Y;\tau) + \frac{1}{2} I^{\mathrm{A}}(2;\tau)$. It follows from Proposition 3.1.8 that the product of two A-elliptic zeta values is again a rational multiple of an A-elliptic zeta value. Thus

$$\mathcal{I}^{\mathrm{A}}(X,Y;\tau) + \mathcal{I}^{\mathrm{A}}(Y,X;\tau) \equiv 0 \mod \mathcal{L}_1(\mathcal{EZ}^{\mathrm{A}}), \tag{4.32}$$

$$\mathcal{I}^{\mathrm{A}}(X,Y;\tau) + \mathcal{I}^{\mathrm{A}}(X+Y,-Y;\tau) + \mathcal{I}^{\mathrm{A}}(-X-Y,X;\tau) \equiv 0 \mod \mathcal{L}_1(\mathcal{EZ}^{\mathrm{A}}). \tag{4.33}$$

Since these two equations are precisely the defining equations of the Fay-shuffle space, the natural surjection

$$\begin{aligned} (\widehat{V}_{k-2})^* &\to \mathrm{gr}_2^{\mathcal{L}} \mathcal{EZ}_k^{\mathrm{A}} \\ (X^{r-1}Y^{s-1})^* &\mapsto I^{\mathrm{A}}(r,s;\tau) \mod \mathcal{L}_1(\mathcal{EZ}_k^{\mathrm{A}}) \end{aligned} \tag{4.34}$$

factors through the annihilator $\mathrm{FSh}_2(k-2)^0 \subset (\widehat{V}_{k-2})^*$ of the subspace $\mathrm{FSh}_2(k-2)$, and therefore

$$\dim_{\mathbb{Q}} \mathrm{gr}_2^{\mathcal{L}} \mathcal{EZ}_k^{\mathrm{A}} \leq \dim_{\mathbb{Q}} \left[(\widehat{V}_{k-2})^* / (\mathrm{FSh}_2(k-2)^0) \right] = \dim_{\mathbb{Q}} \mathrm{FSh}_2(k-2). \tag{4.35}$$

□

4.4.1 The dimension of the Fay-shuffle space

The goal of this section is to prove the following

Theorem 4.4.4. *We have*

$$\dim_{\mathbb{Q}} \mathrm{FSh}_2(d) = \begin{cases} 0, & \text{if } d \text{ is even} \\ \left\lfloor \frac{d+2}{3} \right\rfloor + 1, & \text{if } d \text{ is odd.} \end{cases} \tag{4.36}$$

For even d, the theorem follows from Proposition 4.4.2. For odd d, the proof is divided into two propositions. First, if d is odd, the space $\mathrm{FSh}_2(d)$ splits into a polynomial part, and a non-polynomial part as follows.

Proposition 4.4.5. *For odd $d \geq -1$, we have a short exact sequence*

$$0 \longrightarrow \mathrm{FSh}_2(d)^{pol} \longrightarrow \mathrm{FSh}_2(d) \longrightarrow \mathbb{Q} \longrightarrow 0, \tag{4.37}$$

where $\mathrm{FSh}_2(d)^{pol} = \mathrm{FSh}_2(d) \cap \mathbb{Q}[X,Y]$ *denotes the polynomial part of the Fay-shuffle space and the map on the right is given by*

$$P \mapsto \textit{Coeff. of } \tfrac{X^{d+1}}{Y} \textit{ in } P. \tag{4.38}$$

A splitting is given by mapping $1 \in \mathbb{Q}$ *to*

$$\widetilde{P}(X,Y) = \frac{X^{d+1}}{Y} - \frac{Y^{d+1}}{X} - \frac{X^{d+1} - Y^{d+1}}{X+Y}. \tag{4.39}$$

Proof: That $\widetilde{P}$ satisfies the Fay-shuffle equations is seen by a direct computation (here we use that d is odd). To show exactness, note that from the definition of $\mathrm{FSh}_2(d)$, an element $P(X,Y) \in \mathrm{FSh}_2(d) \setminus \mathrm{FSh}_2(d)^{pol}$ necessarily has the form

$$a\frac{X^{d+1}}{Y} + b\frac{Y^{d+1}}{X} + Q(X,Y), \tag{4.40}$$

where $Q(X,Y) \in \mathbb{Q}[X,Y]$ is some specific polynomial, and $a, b \in \mathbb{Q}$ are not both equal to zero. Hence, we have

$$\dim_{\mathbb{Q}} \mathrm{FSh}_2(d)^{pol} + 1 \leq \dim_{\mathbb{Q}} \mathrm{FSh}_2(d) \leq \dim_{\mathbb{Q}} \mathrm{FSh}_2(d)^{pol} + 2. \tag{4.41}$$

But from the shuffle equation $P(X,Y) + P(Y,X) = 0$, one sees that in (4.40), it holds that $a = -b$, and thus $\dim_{\mathbb{Q}} \mathrm{FSh}_2(d) = \dim_{\mathbb{Q}} \mathrm{FSh}_2(d)^{pol} + 1$. ☐

In order to prove Theorem 4.4.4, it thus suffices to compute the dimension of $\mathrm{FSh}_2(d)^{pol}$, in the case where $d \geq 1$ is odd.

Theorem 4.4.6. *For $d \geq 1$ odd, we have*

$$\dim_{\mathbb{Q}} \mathrm{FSh}_2(d)^{pol} = \left\lfloor \frac{d+2}{3} \right\rfloor. \tag{4.42}$$

The proof of this result will occupy the rest of this subsection.

Proof:[2] We will need some elements of the representation theory of the symmetric group S_3 and also some invariant theory. Possible references are [71, 73]. Denote by $V_d \subset \mathbb{Q}[X,Y]$

[2]The proof given here, which simplifies the author's original proof, was communicated by Francis Brown.

the subspace of homogeneous polynomials of degree d. The group $\mathrm{GL}_2(\mathbb{Q})$ acts on V_d on the right by

$$P(X,Y)_{|M} = P(aX+bY, cX+dY), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.43}$$

Using this action, we turn V_d into an S_3-representation by defining a morphism

$$\begin{gathered} S_3 \to \mathrm{GL}_2(\mathbb{Q}) \\ (12) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \end{gathered} \tag{4.44}$$

From the classification of the irreducible representations of S_3, we see that V_1 is isomorphic to the unique, irreducible two-dimensional S_3-representation U (cf. [71], 2.5). Its character χ_1 is given by

$$\chi_1(1) = 2, \quad \chi_1(12) = 0, \quad \chi_1(123) = -1. \tag{4.45}$$

Furthermore, $V_d \cong \mathrm{Sym}^d U$ as S_3-representations, and the generating series

$$\chi := \sum_{d\geq 0} \chi_d t^d \tag{4.46}$$

of the characters χ_d of V_d is given by (cf. [73], Theorem 2.1)

$$\chi(1) = \frac{1}{(1-t)^2}, \quad \chi(12) = \frac{1}{1-t^2}, \quad \chi(123) = \frac{1}{1+t+t^2}. \tag{4.47}$$

Consider now a decomposition

$$V_d = \bigoplus_{i=1}^{n} W_i \tag{4.48}$$

of V_d into irreducible S_3-subrepresentations of V_d, and denote by u_d the number of the W_i, which are isomorphic to U.

Proposition 4.4.7. *For every d, the number u_d does not depend on the choice of decomposition, and is given explicitly by the generating series*

$$\sum_{d\geq 0} u_d t^d = \frac{t}{(1-t)^2(1+t+t^2)}. \tag{4.49}$$

Proof: It is well-known that u_d is independent of the decomposition, and that it is given by the scalar product of χ_1 and χ_d (cf. [71], 2.3). Computing the scalar product by means of the generating series (4.47), we get the result. □

Corollary 4.4.8. *We have $u_d = \left\lfloor \frac{d+2}{3} \right\rfloor$.*

Using this result, Theorem 4.4.6 follows from

Proposition 4.4.9. *If $d \geq 1$ is odd, then* $\dim_{\mathbb{Q}} \mathrm{FSh}_2(d)^{pol} = u_d$.

Proof: Recall that $\mathrm{FSh}_2(d)^{pol} \subset V_d$ is defined as the space of antisymmetric polynomials, which satisfy the Fay relation

$$P(X,Y) + P(X+Y,-Y) + P(-X-Y,X) = 0 \tag{4.50}$$

But if d is odd, then every $P \in V_d$ satisfying the Fay relation is automatically antisymmetric. Indeed, by applying (4.50) to the polynomial $P(-Y,-X)$, we get

$$P(-Y,-X) + P(-X-Y,X) + P(X+Y,-Y) = 0, \tag{4.51}$$

and hence, by comparing with (4.50), $P(X,Y) = P(-Y,-X) = -P(Y,X)$.
The Fay relation (4.50) can be written using the S_3-action defined in (4.44) as

$$P(X,Y) + P(X,Y)^{(12)} + P(X,Y)^{(123)} = 0. \tag{4.52}$$

Therefore, every non-zero $P \in \mathrm{FSh}_2(d)$ spans a two-dimensional subrepresentation

$$U_P = \mathrm{Span}_{\mathbb{Q}}\{P(X,Y), P(X,Y)^{(12)}\} \subset V_d, \tag{4.53}$$

which is isomorphic to U. Clearly, if $P' = \lambda P$ with some $\lambda \in \mathbb{Q}^\times$, then $U_P = U_{P'}$.
Conversely, given a subrepresentation $W_i \subset V_d$, which is isomorphic to U, we claim that there exists $w \in W_i \setminus \{0\}$, unique up to a non-zero scalar, such that

$$w + w^{(12)} + w^{(123)} = 0. \tag{4.54}$$

Assuming this for a moment, we see that the assignments $P \mapsto U_P$ and $W_i \mapsto w$ are inverse to each other. In particular, the dimension of $\mathrm{FSh}_2(d)$ equals the number of copies of U in (4.48), and the proposition follows.
To prove the claim, we may realize the irreducible two-dimensional S_3-representation U as the hyperplane

$$U = \{(x,y,z) \in \mathbb{Q}^3 \mid x+y+z = 0\}, \tag{4.55}$$

with basis e_1, e_2, where e_i is the i-th unit vector, and the action is given by $(e_i)^\sigma = e_{\sigma^{-1}(i)}$.
By (4.54), and since (123) acts with trace -1 (again by the classification of irreducible S_3-representations), the action of (123) on e_1, e_2 is given by

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.56}$$

Therefore, if $u \in U$ satisfies $u + u^{(12)} + u^{(123)} = 0$, then u must be a multiple of e_1. □

This ends the proof of Theorem 4.4.6. □

4.5 A lower bound for the space of elliptic double zeta values

We use the differential equation (Theorem 3.3.1) to prove a linear independence result for A-elliptic double zeta values.

Theorem 4.5.1. *Let $k > 0$ be odd, and $n = \lfloor \frac{k}{3} \rfloor$.*

(i) *The set*

$$\left\{ \frac{\partial}{\partial \tau} I^{\mathrm{A}}(r, k-r; \tau) \mid 0 \le r \le n \right\} \tag{4.57}$$

is linearly independent over $\mathbb{Q}$.

(ii) *We have*

$$\dim_{\mathbb{Q}} \mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}}) \ge n+1. \tag{4.58}$$

Proof: First note that i) implies ii), as the A-elliptic double zeta values $I^{\mathrm{A}}(r, k-r; \tau)$ for $r = 0, \dots, n$ are linearly independent, since, by i), their derivatives are.
So assume there exists a relation

$$\sum_{r=0}^{n} \lambda_r \frac{\partial}{\partial \tau} I^{\mathrm{A}}(r, k-r; \tau) = 0. \tag{4.59}$$

with $\lambda_r \in \mathbb{Q}$ for $r = 0, \dots, n$. Substituting the differential equation for elliptic double zeta values (Proposition 4.1.1) and multiplying both sides by $2\pi i$, we obtain[3]

$$\begin{aligned} 0 = \lambda_0(kG_0(\tau)I^{\mathrm{A}}(k+1) - kG_{k+1}(\tau)I^{\mathrm{A}}(0)) + \sum_{r=1}^{n} \lambda_r \Big(& -(k-r)G_{k-r+1}(\tau)I^{\mathrm{A}}(r) \\ & + rG_{r+1}(\tau)I^{\mathrm{A}}(k-r) - (-1)^r N G_{k+1}(\tau) I^{\mathrm{A}}(0) \\ & + \sum_{s=1}^{k+1}(k-s)\left(\binom{s-1}{r-1} - \binom{s-1}{k-r-1} \right) G_{k+1-s}(\tau) I^{\mathrm{A}}(s) \Big). \end{aligned} \tag{4.60}$$

Since the Eisenstein series are linearly independent over $\mathbb{C}$, and since the $I^{\mathrm{A}}(2j)$ are linearly independent over $\mathbb{Q}$ (they are non-zero multiples of $(2\pi i)^{2j}$, cf. Proposition 3.1.8), it follows that the family $I^{\mathrm{A}}(2j)G_{k+1-2j}$ is linearly independent over $\mathbb{Q}$. Thus, by (4.60) there exist $\lambda_0, \dots, \lambda_n$ as above, if and only if the row vector $\Lambda_k = (\lambda_0, \dots, \lambda_n) \in \mathbb{Q}^{n+1}$ solves the linear system of equations

$$\Lambda_k \cdot C_k = (0, \dots, 0), \tag{4.61}$$

[3]Here and in the following, for typographical reasons, we shall write $I^{\mathrm{A}}(k)$ instead of $I^{\mathrm{A}}(k;\tau)$, i.e. we suppress the τ-dependence (also note that $I^{\mathrm{A}}(k;\tau)$ does not depend on τ, by Proposition 3.1.8).

where C_k is the $(n+1)\times(k+3)/2$ matrix whose entry $(C_k)_{r,s}$ is given by

$$(C_k)_{r,s} = \text{Coefficient of } \lambda_r I^{\mathrm{A}}(2s)G_{k+1-2s}(\tau) \text{ in (4.60)}, \quad 0\le r\le n,\ 0\le s\le \frac{k+1}{2}. \tag{4.62}$$

Hence, if we can prove that the rank of C_k is $n+1$, we are done, because then (4.61) has only the trivial solution $\Lambda_k = (0,\dots,0)$. Also, note that the first row of C_k is equal to $(-k,0,\dots,0,k)$ by (4.60). Therefore, if we can prove that among the columns of C_k indexed by $s=1,\dots,(k-1)/2$, there are n linearly independent ones, C_k will have rank exactly $n+1$. For this, we can clearly assume that $k\ge 3$, for there are no columns left otherwise, and the Theorem would be trivially true.
To this end, consider, for $k\ge 3$ and k odd, the square submatrix C_k' of C_k consisting of the rows $r=1,\dots,n$ and the columns $s=1,\dots,n$. Looking at (4.60) , we see that its entries are given by

$$(C_k')_{i,j} = (k-2j-2)\left(\binom{2j+1}{i} - \delta_{2j+1,i}\right), \quad 0\le i,j\le n-1. \tag{4.63}$$

Since $k-2j-2\neq 0$ for every j, as k is odd, it is enough to prove that the scaled matrix $M_{n-1}=(m_{i,j})$ with

$$m_{i,j} = \binom{2j+1}{i} - \delta_{2j+1,i}, \quad 0\le i,j\le n-1 \tag{4.64}$$

is invertible for every $n\ge 1$. This is proved in Section 4.8, Proposition 4.8.1. □

4.6 The main result on A-elliptic double zeta values

The culmination of the work of the preceding subsections is the following

Theorem 4.6.1. (i) *Let $k\ge 0$ and $D^{ell}_{k,2} := \mathrm{gr}_2^{\mathcal{L}}\mathcal{E}\mathcal{Z}_k^{\mathrm{A}}$. Then*

$$D^{ell}_{k,2} = \begin{cases} 0, & \text{if } k \text{ is even} \\ \left\lfloor \dfrac{k}{3}\right\rfloor + 1, & \text{if } k \text{ is odd.} \end{cases} \tag{4.65}$$

(ii) *Every $\mathbb{Q}$-linear relation between length-graded elliptic double zeta values is a consequence of Fay and shuffle relations.*

Proof: From Theorem 4.4.4, we know that, if k is even

$$\dim_{\mathbb{Q}} \mathrm{gr}_2^{\mathcal{L}}\mathcal{E}\mathcal{Z}_k^{\mathrm{A}} \le \dim_{\mathbb{Q}} \mathrm{FSh}_2(k-2) = 0, \tag{4.66}$$

which proves Theorem 4.6.1 in that case. For odd k, consider again the surjection (cf. Proposition 4.4.3)

$$\begin{aligned}(\widehat{V}_{k-2})^*/\operatorname{FSh}_2(k-2)^0 &\to \operatorname{gr}_2^{\mathcal{L}}\mathcal{EZ}_k^{\mathrm{A}}\\ (X^*)^{r-1}(Y^*)^{s-1} &\mapsto I^{\mathrm{A}}(r,s;\tau) \mod \mathcal{L}_1(\mathcal{EZ}_k^{\mathrm{A}}).\end{aligned} \tag{4.67}$$

By Theorem 4.4.4, the left hand side has dimension $\left\lfloor \frac{k}{3}\right\rfloor + 1$. On the other hand, by Theorem 4.5.1, the elliptic double zeta values

$$I^{\mathrm{A}}(r,k-r;\tau) \quad 0 \le r \le \left\lfloor \frac{k}{3}\right\rfloor \tag{4.68}$$

are linearly independent over $\mathbb{Q}$. Since $\mathcal{L}_1(\mathcal{EZ}_k^{\mathrm{A}}) = 0$ if k is odd (cf. Proposition 3.1.8), we get

$$\dim_{\mathbb{Q}} \operatorname{gr}_2^{\mathcal{L}}\mathcal{EZ}_k^{\mathrm{A}} \ge \left\lfloor \frac{k}{3}\right\rfloor + 1. \tag{4.69}$$

Thus, (4.67) is an isomorphism, showing that $\dim_{\mathbb{Q}} \operatorname{gr}_2^{\mathcal{L}}\mathcal{EZ}_k^{\mathrm{A}} = \left\lfloor \frac{k}{3}\right\rfloor + 1$. Also, since there are no non-trivial $\mathbb{Q}$-linear relations between elliptic double zeta values of different weights (cf. Theorem 4.2.1), the isomorphism (4.67) also shows that all $\mathbb{Q}$-linear relations between length-graded elliptic double zeta values are a consequence of Fay and shuffle relations. □

4.7 A partial result in length three

In this chapter, we have proved a formula for $D^{ell}_{k,2}$ for all $k \ge 0$, using linear independence results on A-elliptic double zeta values as well as representation theory of finite groups. In principle, there seems to be no conceptual bottleneck in extending the computation of $D^{ell}_{k,n}$ from $n = 2$ to all n. However, there are a few technical obstacles to surmount. For example, the Fay-shuffle relations, whose study played a crucial role in this Chapter, become more complicated in higher lengths. In fact, the Fay-shuffle relations are not entirely well-defined for all lengths, and this will be the subject of an ongoing joint work with P. Lochak and L. Schneps [56]. Furthermore, even if one had the Fay-shuffle relations in all lengths, the representation theory involved in computing the associated Fay-shuffle spaces is likely to become more involved. A similar picture emerged in the study of multiple zeta values, where one has results in depths[4] two [83] and three [40], but not in higher depth. For these reasons, we contend ourselves with giving a partial result for $n = 3$.

Theorem 4.7.1. *For k odd, we have*

$$\dim_{\mathbb{Q}} \operatorname{gr}_3^{\mathcal{L}}(\mathcal{EZ}_k^{\mathrm{A}}) = \left\lfloor \frac{k+1}{6}\right\rfloor. \tag{4.70}$$

[4]Recall that the length of an A-elliptic multiple zeta value is the analogue of the depth of a multiple zeta value.

Proof: Since $\dim_{\mathbb{Q}} \mathrm{gr}_3^{\mathcal{L}}(\mathcal{EZ}_k^{\mathrm{A}}) = \dim_{\mathbb{Q}} \mathcal{L}_3(\mathcal{EZ}_k^{\mathrm{A}}) - \dim_{\mathbb{Q}} \mathcal{L}_2(\mathcal{EZ}_k^{\mathrm{A}}) = \dim_{\mathbb{Q}} \mathcal{L}_3(\mathcal{EZ}_k^{\mathrm{A}}) - \left\lfloor \frac{k}{3} \right\rfloor - 1$ by Theorem 4.6.1 (note that $\mathcal{L}_1(\mathcal{EZ}_k^{\mathrm{A}}) = \{0\}$, if k is odd, which follows from Proposition 3.1.8), it suffices to prove that $\dim_{\mathbb{Q}} \mathcal{L}_3(\mathcal{EZ}_k^{\mathrm{A}}) = \frac{k+1}{6} + \left\lfloor \frac{k}{3} \right\rfloor + 1 = \frac{k+1}{2}$.
In order to see this, note that by the shuffle product formula for A-elliptic multiple zeta values (3.10), every $I^{\mathrm{A}}(r;\tau)I^{\mathrm{A}}(0,s;\tau)$ with r even and s odd is an A-elliptic multiple zeta value of length three and weight $k := r+s$. These are linearly independent over $\mathbb{Q}$, since $I^{\mathrm{A}}(0,s;\tau) = \frac{1}{2\pi i}\mathcal{G}^0(s+1;\tau) + \frac{\pi i}{2}\delta_{s,1}$ by Proposition 4.1.1 and the discussion following it. On the other hand, it follows from (3.35) that the differential equation for A-elliptic multiple zeta values in length three is

$$\begin{aligned}
&2\pi i \frac{\partial}{\partial \tau} I^{\mathrm{A}}(k_1,k_2,k_3;\tau) \\
&\quad = k_1 G_{k_1+1}(\tau) I^{\mathrm{A}}(k_2,k_3;\tau) - k_r G_{k_r+1}(\tau) I^{\mathrm{A}}(k_1,k_2;\tau) \\
&\quad + \sum_{i=2}^{3} \Big\{ (-1)^{k_i}(k_{i-1}+k_i) G_{k_{i-1}+k_i+1}(\tau) I^{\mathrm{A}}(k_1,\dots,k_{i-2},0,k_{i+1},\dots,k_3;\tau) \\
&\quad - \sum_{k=0}^{k_{i-1}+1} (k_{i-1}-k)\binom{k_i+k-1}{k} G_{k_{i-1}-k+1}(\tau) I^{\mathrm{A}}(k_1,\dots,k_{i-2},k+k_i,k_{i+1},\dots,k_3;\tau) \\
&\quad + \sum_{k=0}^{k_i+1} (k_i-k)\binom{k_{i-1}+k-1}{k} G_{k_i-k+1}(\tau) I^{\mathrm{A}}(k_1,\dots,k_{i-2},k+k_{i-1},k_{i+1},\dots,k_3;\tau) \Big\} .
\end{aligned} \tag{4.71}$$

If the weight $k_1+k_2+k_3$ is odd, then every term on the right hand side of (4.71) is a $\mathbb{Q}$-linear combination of products $G_r(\tau)I^{\mathrm{A}}(s,t;\tau)$, with $r+s+t$ even. Such a product is non-vanishing, only if r is even, which implies that $s+t$ must be even as well. But in this case, $I^{\mathrm{A}}(s,t;\tau) \in \mathbb{Q}(2\pi i)^{r+s}$, by Proposition 4.1.2.
Now integrating both sides of (4.71), it follows that $I^{\mathrm{A}}(k_1,k_2,k_3;\tau)$ is a $\mathbb{Q}$-linear combination of products

$$(2\pi i)^j \mathcal{G}(k_1+k_2+k_3+1-j;\tau), \tag{4.72}$$

and the constant term $I_0^{\mathrm{A}}(k_1,k_2,k_3;\tau)$ of $I^{\mathrm{A}}(k_1,k_2,k_3;\tau)$. On the other hand, by the results of Section 3.3.2, the constant term is non-vanishing, if and only if $k_1 = 1$ and k_2,k_3 are both even, or $k_3 = 1$ and k_1,k_2 are both even. In the first case, $I_0^{\mathrm{A}}(k_1,k_2,k_3)$ is proportional to $G_2(\tau)I^{\mathrm{A}}(k_2,k_3;\tau)$ and in the second case $I_0^{\mathrm{A}}(k_1,k_2,k_3)$ is proportional to $G_2(\tau)I^{\mathrm{A}}(k_1,k_2;\tau)$. Thus, it follows that every A-elliptic multiple zeta value of length three and odd weight k is a unique $\mathbb{Q}$-linear combination of products

$$I^{\mathrm{A}}(r;\tau)I^{\mathrm{A}}(0,s;\tau), \quad k = r+s, \quad r \text{ even} \tag{4.73}$$

and the number of those is precisely equal to $\frac{k+1}{2}$. □

Theorem 4.7.1 only treats the case of odd weight elliptic multiple zeta values of length three, which can be described using linear combinations of (single) Eisenstein integrals

$\mathcal{G}(2k;\tau)$. Even weight elliptic multiple zeta values involve double Eisenstein integrals, i.e. integrals of the form

$$\int_{\tau}^{i\infty} G_{2k_1}(\tau_1)\mathcal{G}(2k_2;\tau_1)\mathrm{d}\tau_1, \quad k_1,k_2 \geq 0. \tag{4.74}$$

A complication that arises is that, in contrast to single Eisenstein integrals, not every linear combination of double Eisenstein integrals will occur as an A-elliptic (or B-elliptic for that matter) multiple zeta value. More precisely, the linear combinations of iterated Eisenstein integrals, which can possibly occur as elliptic multiple zeta values are intimately connected to relations in a certain Lie algebra derivations. This will be described in detail in Chapter 5.

4.8 A binomial determinant

The proof of Theorem 4.5.1 depends on the invertibility of a certain family of matrices $(M_n)_{n\geq 0}$, the entries of which are given by binomial coefficients. We now complete the proof of Theorem 4.5.1 by computing the determinant of the aforementioned binomial matrices explicitly.

Let n be a non-negative integer and consider the matrix $M_n = (m_{i,j})_{0\leq i,j\leq n}$ with

$$m_{i,j} = \binom{2j+1}{i} - \delta_{2j+1,i}, \tag{4.75}$$

where δ denotes the Kronecker delta. This is precisely the matrix given in (4.64), on whose invertibility the proof of Theorem 4.5.1 relied.

Proposition 4.8.1. *We have*

$$\det(M_n) = (2n+1)!! = 1\cdot 3\cdot 5\ldots\cdot(2n+1). \tag{4.76}$$

In particular, M_n is invertible for every n.

The idea of the proof of Proposition 4.8.1 is to find a suitable LU-decomposition for M_n. We first need a lemma about binomial coefficients.

Lemma 4.8.2. *For all $a,b\geq 0$, we have*

$$\binom{a}{b} = \sum_{k=0}^{b}\binom{a-b+k}{k}\binom{a-b+1}{a-2b+2k+1}. \tag{4.77}$$

Proof: We first assume that $b\leq a/2$. In that case, the right hand side is equal to

$$\binom{a-b+1}{a-2b+1}{}_3F_2\left[\begin{matrix} a-b+1, & -b/2+1/2, & -b/2 \\ & a/2-b+1, & a/2-b+3/2 \end{matrix};1\right], \tag{4.78}$$

where

$$ {}_3F_2\left[{\alpha_1,\ \alpha_2,\ \alpha_3 \atop \rho_1,\ \rho_2};z\right] := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{k!(\rho_1)_k(\rho_2)_k} z^k \tag{4.79} $$

is a hypergeometric function (cf. [4], Chapter II). Here, $(m)_k := m(m+1)\dots(m+k-1)$ denotes the Pochhammer symbol. Now if b is even, we can apply *Saalschütz's Theorem*

$$ {}_3F_2\left[{a,\ b,\qquad -n \atop c,\ a+b-c+1-n};1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \tag{4.80} $$

(cf. [4], Section 2.2). Hence, (4.78) is equal to

$$ \binom{a-b+1}{a-2b+1}\frac{(-a/2)_{b/2}(a/2-b/2+1/2)_{b/2}}{(a/2-b+1)_{b/2}(-a/2+b/2-1/2)_{b/2}} = \binom{a}{b}. \tag{4.81} $$

In the case where b is odd, we can apply Saalschütz's Theorem after interchanging $-b/2$ and $-b/2+1/2$ in the argument of ${}_3F_2$ in (4.78) above, and get the result. If $b > a/2$, then the same argument as above works, with b replaced by $a-b$ throughout. □

Now consider the matrices $L_n = (l_{i,j})_{0\le i,j\le n}$, $U_n = (u_{i,j})_{0\le i,j\le n}$ with

$$ l_{i,j} = \binom{j}{i-j}, \quad u_{i,j} = \begin{cases} 1, & i=0 \\ \binom{2j-i}{i-1}\dfrac{2j+1}{i}, & 0<i<2j+1 \\ 0, & i\ge 2j+1. \end{cases} \tag{4.82} $$

Note that L_n is a lower triangular matrix with determinant 1, while U_n is an upper triangular matrix with determinant $(2n+1)!!$. Hence, Proposition 4.8.1 follows from

Lemma 4.8.3. *For every n, we have*

$$ M_n = L_n U_n. \tag{4.83} $$

Proof: We see from (4.82) that the assertion of the lemma is equivalent to

$$ \binom{2j+1}{i} - \delta_{2j+1,i} = \sum_{k=0}^{n} l_{i,k}u_{k,j}, \tag{4.84} $$

for all i,j such that $0 \le i,j \le n$. For $i \ge 2j+1$, both sides of (4.84) are evidently equal to zero. If $i=0$, then, since $l_{0,k} = \delta_{0,k}$, both sides of (4.84) are equal to 1.
It remains to prove (4.84) for i,j such that $0 < i < 2j+1 \le n$. In this case, we have $l_{i,0} = 0$, and since $\binom{k}{i-k}$ vanishes for $k > i$, we need to prove that

$$ \binom{2j+1}{i} - \delta_{2j+1,i} = \sum_{k=1}^{i} \binom{k}{i-k}\binom{2j-k}{k-1}\frac{2j+1}{k}. \tag{4.85} $$

We now rewrite the right hand side of (4.85) as

$$\begin{aligned}
\sum_{k=1}^{i} \binom{k}{i-k}\binom{2j-k}{k-1}\frac{2j+1}{k} &= \sum_{k=0}^{i} \binom{k}{i-k}\binom{2j+1-k}{k}\frac{2j+1}{2j+1-k} \\
&= \sum_{k=0}^{i} \binom{2j+1-k}{i-k}\binom{2j+1-i}{2k-i}\frac{2j+1}{2j+1-k} \\
&= \sum_{k=0}^{i} \binom{2j+1-k}{2j+1-i}\binom{2j+1-i}{2j-2k+1}\frac{2j+1}{2j+1-k} \\
&= \sum_{k=0}^{i} \binom{2j-k}{2j-i}\binom{2j+1-i}{2j-2k+1}\frac{2j+1}{2j+1-i} \\
&= \sum_{k=0}^{i} \binom{2j-k}{i-k}\binom{2j+1-i}{2j-2k+1}\frac{2j+1}{2j+1-i} \\
&= \sum_{k=0}^{i} \binom{2j-i+k}{k}\binom{2j+1-i}{2j-2i+2k+1}\frac{2j+1}{2j+1-i},
\end{aligned} \tag{4.86}$$

where we freely used standard properties of binomial coefficients. Applying Lemma 4.8.2 with $a = 2j$ and $b = i$ to the last line of (4.86), we finally obtain (4.85), as desired. □

Chapter 5

Elliptic multiple zeta values and iterated Eisenstein integrals

In the last chapter, we have made use of the length filtration to study the space of elliptic multiple zeta values of a fixed length. This led to the computation of the dimension of the space of A-elliptic double zeta values (Theorem 4.6.1). As we have seen, an important tool was the differential equation satisfied by A-elliptic multiple zeta values (Theorem 3.3.1), which relates them to (iterated) integrals of Eisenstein series.
In this section, we study elliptic multiple zeta values from a more "global" point of view, that is, we do not only consider elliptic multiple zeta values of a fixed length, but all elliptic multiple zeta values at the same time. The crucial tool to use is again the differential equation for elliptic multiple zeta values, but studied from a slightly different point of view, which elucidates its relation to a certain Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ of special derivations on the fundamental Lie algebra of a once-punctured elliptic curve, which have been introduced in a slightly different version in [60, 75], and have since then re-appeared in work of many people [7, 22, 24, 31, 45, 61]. More precisely, their differential equation shows that elliptic multiple zeta values are linear combinations of iterated integrals of Eisenstein series [20, 57], whose coefficients are controlled by $\mathfrak{u}^{\mathrm{geom}}$. The classical multiple zeta values make an appearance as boundary conditions for said differential equation. To summarize this chapter, we make a first step towards a complete algebraic description of elliptic multiple zeta values in terms of classical multiple zeta values and iterated Eisenstein integrals.

5.1 Preliminaries

In the previous sections, the generating series of A-elliptic and B-elliptic multiple zeta values $\underline{A}(\tau)$ and $\underline{B}(\tau)$ were series in the non-commutative formal variables x_0 and x_1. In this chapter, however, we will make a global change of variables

$$a := 2\pi i x_0, \quad b := (2\pi i)^{-1} x_1. \tag{5.1}$$

and will rewrite $\underline{A}(\tau)$ and $\underline{B}(\tau)$ as series in a and b. Although it is merely a straightforward substitution, this change of variables will have the effect of eliminating cumbersome powers of $2\pi i$, and will elucidate the structure of elliptic multiple zeta values. For similar reasons, in this chapter we will work with the Hecke-normalized Eisenstein series $E_{2k}(\tau) = \frac{(2k-1)!}{2(2\pi i)^{2k}} G_{2k}(\tau)$, instead of the Eisenstein series $G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^k}$.
We begin with some technical preliminaries on iterated Eisenstein integrals and the above-mentioned Lie algebra of special derivations.

5.1.1 Iterated Eisenstein integrals

The study of iterated integrals of modular forms has been initiated by Manin [57], and was extended further by Brown [20]. In this section, we follow [20], Section 4, with slight modifications.
For $k \geq 1$, let

$$E_{2k}(\tau) := \frac{(2k-1)!}{2(2\pi i)^{2k}} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}} \tag{5.2}$$

denote the Eisenstein series of weight $2k$. It has a Fourier expansion in $q = e^{2\pi i\tau}$, which is given by

$$E_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n\geq 1} \sigma_{2k-1}(n) q^n, \quad \sigma_m(n) = \sum_{d|n} d^m. \tag{5.3}$$

Note that the multiplication by $\frac{(2k-1)!}{2(2\pi i)^{2k}}$ has the effect that the Fourier coefficients of E_{2k} are rational numbers. We also set $E_0 := -1$. Associated to E_{2k}, we have the differential one-form

$$\underline{E}_{2k}(\tau) := 2\pi i \cdot E_{2k}(\tau) \mathrm{d}\tau = E_{2k}(\tau) \frac{\mathrm{d}q}{q}. \tag{5.4}$$

Viewed as a meromorphic differential one-form in the variable q, it is defined over $\mathbb{Q}$.
Now let $\mathbf{e} := \{\mathbf{e}_{2k}\}_{k\geq 0}$ be a set of non-commuting variables, and consider the generating series

$$\omega_{\mathrm{Eis}} := \sum_{k\geq 0} \underline{E}_{2k}(\tau) \cdot \mathbf{e}_{2k} \in \Omega^1(\mathbb{H}) \widehat{\otimes} \mathbb{Q}\langle\langle \mathbf{e} \rangle\rangle, \quad \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \tag{5.5}$$

of differential one-forms of Eisenstein series. We are interested in solutions to the linear differential equation

$$\mathrm{d}f = -\omega_{\mathrm{Eis}} \cdot f, \quad f : \mathbb{H} \to \mathbb{C}\langle\langle \mathbf{e} \rangle\rangle. \tag{5.6}$$

Since for $k \geq 1$, we have $\lim_{\tau\to i\infty} E_{2k}(\tau) = -\frac{B_{2k}}{4k}$, the differential equation (5.6) has a regular singular point at $i\infty$, with residue

$$\mathrm{res}_\infty := \mathbf{e}_0 + \sum_{k\geq 0} \frac{B_{2k}}{4k} \cdot \mathbf{e}_{2k}. \tag{5.7}$$

Using Picard iteration (cf. e.g. [46], Section 2), it follows that for fixed $\rho \in \mathbb{H}$, the series[1]

$$\exp\left[\int_\tau^\rho \omega_{\mathrm{Eis}}\right] := 1 + \sum_{k\geq 1} \int_\tau^\rho \omega_{\mathrm{Eis}}^k \tag{5.8}$$

satisfies (5.6). Furthermore, we have

$$\exp\left[\int_\tau^\rho \omega_{\mathrm{Eis}}\right] \sim \exp(-\rho \cdot \mathrm{res}_\infty), \quad \text{for } \rho \to i\infty \tag{5.9}$$

Hence, by Proposition A.2.6, the limit

$$\mathcal{E}(\tau) := \lim_{\rho\to i\infty} \exp\left[\int_\tau^\rho \omega_{\mathrm{Eis}}\right] \exp(\rho \cdot \mathrm{res}_\infty) \tag{5.10}$$

exists, and is also a solution to (5.6).

Definition 5.1.1. For a multi-index $\underline{2k} = (2k_1, \dots, 2k_n) \in (2\mathbb{Z}_{\geq 0})^n$, define the *regularized iterated Eisenstein integral*

$$\mathcal{E}(\underline{2k}; \tau) \in \mathcal{O}(\mathbb{H}) \tag{5.11}$$

to be the coefficient of $\mathbf{e}_{\underline{2k}} := \mathbf{e}_{2k_1} \dots \mathbf{e}_{2k_n}$ in $\mathcal{E}(\tau)$.
Furthermore, denote by $\langle\mathcal{E}\rangle_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space spanned by the iterated Eisenstein integrals.

Proposition 5.1.2. *The $\mathbb{Q}$-vector space $\langle\mathcal{E}\rangle_{\mathbb{Q}}$ is a $\mathbb{Q}$-algebra.*

Proof: It follows from Proposition A.2.3 that the series

$$\exp\left[\int_\tau^\rho \omega_{\mathrm{Eis}}\right] \tag{5.12}$$

is group-like. Therefore, also $\mathcal{E}(\tau)$ as defined in 5.10 is group-like, and by Corollary A.1.8 the $\mathbb{Q}$-vector space spanned by its coefficients, which is just $\langle\mathcal{E}\rangle_{\mathbb{Q}}$ is a $\mathbb{Q}$-algebra. □

In order to describe the structure of $\langle\mathcal{E}\rangle_{\mathbb{Q}}$, recall the definition of the tensor $\mathbb{Q}$-algebra $T(\mathbf{e})$ on the set $\mathbf{e}$. As a $\mathbb{Q}$-vector space, $T(\mathbf{e})$ is given by the direct sum

$$T(\mathbf{e}) = \bigoplus_{n\geq 0} \langle\mathbf{e}\rangle_{\mathbb{Q}}^{\otimes n}, \quad \langle\mathbf{e}\rangle_{\mathbb{Q}} := \mathrm{Span}_{\mathbb{Q}}\{\mathbf{e}_{2k} \,|\, k \geq 0\}, \tag{5.13}$$

and the multiplication is given by concatenation of tensors. This defines an associative, unital multiplication law on $T(\mathbf{e})$, thus gives $T(\mathbf{e})$ the structure of a $\mathbb{Q}$-algebra. Moreover, $T(\mathbf{e})$ is a graded $\mathbb{Q}$-algebra, where elements of $\langle\mathbf{e}\rangle_{\mathbb{Q}}^{\otimes n}$ have degree n. We will write $T(\mathbf{e})_n$

[1]Since $\mathbb{H}$ is a simply connected Riemann surface and ω_{Eis} is a holomorphic differential one-form, the iterated integral does not depend on the choice of path from τ to ρ.

instead of $\langle\mathbf{e}\rangle_{\mathbb{Q}}^{\otimes n}$, and we note that $T(\mathbf{e})_n$ is finite-dimensional for every n. Denote by $T(\mathbf{e})^\vee$ the graded dual of $T(\mathbf{e})$

$$T(\mathbf{e})^\vee = \bigoplus_{n\geq 0} T(\mathbf{e})_n^\vee, \tag{5.14}$$

i.e. the direct sum of the duals of the degree n components of $T(\mathbf{e})$. The $\mathbb{Q}$-vector space $T(\mathbf{e})^\vee$ carries a natural structure of a graded $\mathbb{Q}$-algebra, whose product is given by the shuffle product. In fact, with the notation of Appendix A.1, we have $T(\mathbf{e})^\vee \cong \mathbb{Q}\langle\mathbf{e}\rangle$. Elements of $T(\mathbf{e})^\vee$ will be written as linear combinations of dual elements $\mathbf{e}_{2k_1}^\vee \dots \mathbf{e}_{2k_n}^\vee$, where $\mathbf{e}_{2k}^\vee(\mathbf{e}_{2l}) = \delta_{k,l}$ and likewise for products of $\mathbf{e}_{2k_i}^\vee$'s.

Theorem 5.1.3. *Let $K \subset \mathbb{C}$ be a subfield. Then $\langle\mathcal{E}\rangle_K$ is a free shuffle algebra. More precisely, the morphism*

$$\begin{aligned} \Psi : \langle\mathcal{E}\rangle_K &\to T(\mathbf{e})^\vee \otimes K \\ \mathcal{E}(\underline{2k};\tau) &\mapsto \mathbf{e}_{2k_1}^\vee \dots \mathbf{e}_{2k_n}^\vee, \end{aligned} \tag{5.15}$$

is a well-defined isomorphism of K-algebras. In particular, $\langle\mathcal{E}\rangle_K$ is graded for the length of iterated integrals:

$$\langle\mathcal{E}\rangle_K = \bigoplus_{n\geq 0} \mathcal{L}_n(\langle\mathbf{e}\rangle_K), \tag{5.16}$$

where, for $n \geq 0$, we set $\mathcal{L}_n(\langle\mathbf{e}\rangle_K) := \mathrm{Span}_K\{\mathcal{E}(2k_1,\dots,2k_n;\tau) \,|\, k_i \geq 0\}$.

Proof: First, the morphism Ψ is well-defined, since the $\mathcal{E}(\underline{2k};\tau)$ are linearly independent over $\mathbb{C}$ (and hence over every subfield of $\mathbb{C}$) by [58], and Ψ is a homomorphism of K-algebras since both sides are endowed with the shuffle product. Moreover, since the elements $\mathbf{e}_{2k_1}^\vee \dots \mathbf{e}_{2k_n}^\vee$ for $k_i \geq 0$ form a basis of $T(\mathbf{e})^\vee$ (by the universal property of $T(\mathbf{e})$), it follows that Ψ is an isomorphism of K-algebras. □

We conclude this section with some closed formulas for iterated Eisenstein integrals in lengths one and two, which are taken from [20], Example 4.10,.

Example 5.1.4. For $k \geq 0$, write

$$E_{2k}^\infty = -\frac{B_{2k}}{4k}, \quad E_{2k}^0(\tau) = \begin{cases} \sum_{n\geq 1} \sigma_{2k-1}(n)q^n & k > 0 \\ 0 & k = 0, \end{cases} \tag{5.17}$$

and let $\underline{E}_{2k}^\infty = 2\pi i \cdot E_{2k}^\infty(\tau)\mathrm{d}\tau$ and $\underline{E}_{2k}^0 = 2\pi i \cdot E_{2k}^0(\tau)\mathrm{d}\tau$ be the associated differential one-forms. Then, we have

$$\mathcal{E}(2k;\tau) = \lim_{\rho\to i\infty}\Big[-\underline{E}_{2k}^\infty \cdot \rho + \int_\tau^\rho \underline{E}_{2k}(\tau_1)\Big] = -\underline{E}_{2k}^\infty \cdot \tau + \int_\tau^{i\infty} \underline{E}_{2k}^0(\tau_1). \tag{5.18}$$

and

$$\begin{aligned}\mathcal{E}(2k_1, 2k_2; \tau) = \int_\tau^{i\infty} \underline{E}_{2k_1}(\tau_1)\underline{E}^0_{2k_2}(\tau_2) - \int_\tau^{i\infty} \underline{E}^\infty_{2k_2}(\tau_1)\underline{E}^0_{2k_1}(\tau_2)\\ - \int_\tau^{i\infty} \underline{E}^0_{2k_1}(\tau_1)\int_0^\tau \underline{E}^\infty_{2k_2}(\tau_1) + \int_0^\tau \underline{E}^\infty_{2k_2}(\tau_1)\underline{E}^\infty_{2k_1}(\tau_2).\end{aligned} \tag{5.19}$$

Remark 5.1.5. In general, $\mathcal{E}(2k_1, \dots, 2k_n; \tau)$ will have an expansion in q and $\log(q) := 2\pi i\tau$ with rational coefficients. More precisely,

$$\mathcal{E}(2k_1, \dots, 2k_n; \tau) \in \mathbb{Q}[[q]] \otimes_\mathbb{Q} \mathbb{Q}[\log(q)] \tag{5.20}$$

for all $k_i \geq 0$.

5.1.2 Special derivations

In this section, we study a family derivations ε_{2k} on a free Lie algebra on two generators. These derivations generate a Lie algebra $\mathfrak{u}^{\mathrm{geom}}$, which first occurred in a slightly different form in work of Nakamura [60] and Tsunogai [75] on Galois representations of once-punctured elliptic curves. More recently, the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ and various versions thereof re-appeared in a variety of articles [7, 22, 24, 31, 45, 61, 69]. We will freely use some basic notions from the theory of Lie algebras, which are collected in Appendix A.
Let $\mathcal{L}$ be the free Lie algebra on the set $\{a, b\}$ [71]. By definition, every derivation on $\mathcal{L}$ is uniquely determined by its values on a and b.

Definition 5.1.6. For every $k \geq 0$, define a derivation $\varepsilon_{2k} : \mathcal{L} \to \mathcal{L}$ by

$$\varepsilon_{2k}(a) = \mathrm{ad}^{2k}(a)(b) \tag{5.21}$$

$$\varepsilon_{2k}(b) = \sum_{0\leq n<k} (-1)^n [\mathrm{ad}^n(a)(b), \mathrm{ad}^{2k-1-n}(a)(b)]. \tag{5.22}$$

Define $\mathfrak{u}^{\mathrm{geom}} \subset \mathrm{Der}(\mathcal{L})$ to be the Lie subalgebra generated by the ε_{2k}.

The Lie algebra $\mathcal{L}$ is bi-graded by giving a bi-degree $(1, 0)$ and b bi-degree $(0, 1)$. We denote by $\mathcal{L}_{(m,n)}$ the homogeneous component of bi-degree (m, n).

Proposition 5.1.7. *The derivations ε_{2k} are uniquely determined by the following three conditions:*

(i) $\varepsilon_{2k}(a) = \mathrm{ad}^{2k}(a)(b)$

(ii) $\varepsilon_{2k}([a, b]) = 0$

(iii) ε_{2k} *is homogeneous of bi-degree* $(2k - 1, 1)$

Proof: That the derivations ε_{2k} satisfy i) and iii) follows directly from their definition, while property ii) is proved in [61], Section 3. Conversely, assume that there exists a derivation δ_{2k} satisfying i)-iii). In particular, we have

$$0 = \delta_{2k}([a,b]) = [\delta_{2k}(a), b] + [a, \delta_{2k}(b)], \tag{5.23}$$

thus $[a, \delta_{2k}(b)]$ is uniquely determined by $\delta_{2k}(a)$. Now $\delta_{2k}(a)$ equals $\varepsilon_{2k}(a)$ by assumption, and it follows that $[a, \delta_{2k}(b)] = [a, \varepsilon_{2k}(b)]$. Since $\mathcal{L}$ is free, the commutator of a in $\mathcal{L}$ is given by multiples of a, and therefore

$$\delta_{2k}(b) = \varepsilon_{2k}(b) + \lambda \cdot a, \tag{5.24}$$

for some $\lambda \in \mathbb{Q}$. By homogeneity of δ_{2k}, we have $\lambda = 0$, and therefore, $\varepsilon_{2k}(b) = \delta_{2k}(b)$. Thus, the derivations ε_{2k} and δ_{2k} agree on a and on b, hence they are identically equal. □

Corollary 5.1.8. *The map "evaluation at a"*

$$\begin{aligned} \psi_a : \mathfrak{u}^{\mathrm{geom}} &\to \mathcal{L} \\ \delta &\mapsto \delta(a), \end{aligned} \tag{5.25}$$

is injective. On the other hand, the map "evaluation at b"

$$\begin{aligned} \psi_b : \mathfrak{u}^{\mathrm{geom}} &\to \mathcal{L} \\ \delta &\mapsto \delta(b), \end{aligned} \tag{5.26}$$

has a one-dimensional kernel given by $\mathbb{Q} \cdot \varepsilon_0$.

Proof: The Lie bracket of two homogeneous derivations of bi-degrees (k,l) and (k',l') is again homogeneous of bi-degree $(k+k', l+l')$ and maps the commutator $[a,b]$ to zero. By Proposition 5.1.7, it follows that every element of $\mathfrak{u}^{\mathrm{geom}}$ is uniquely determined by its value on a, which proves injectivity of (5.25). On the other hand, the equality $\delta(b) = 0$ combined with the equality

$$0 = \delta([a,b]) = [\delta(a), b] + [a, \delta(b)], \tag{5.27}$$

implies $[\delta(a), b] = 0$. Since $\mathcal{L}$ is the free Lie algebra on a and b, it follows that $\delta(a) \in \mathbb{Q} \cdot b$, which, by homogeneity of δ, is only possible if $\delta = \lambda \varepsilon_0$ for $\lambda \in \mathbb{Q}$. □

The preceding proposition implies in particular that the derivations ε_{2k} are uniquely determined by their action on a.

Later on, we will also need a variant $\widetilde{\varepsilon}_{2k}$ of ε_{2k}, defined by

$$\widetilde{\varepsilon}_{2k} = \begin{cases} \dfrac{2}{(2k-2)!}\varepsilon_{2k} & k > 0 \\ \\ -\varepsilon_0 & k = 0. \end{cases} \tag{5.28}$$

5.2 The differential equation for the elliptic KZB associator

The elliptic KZB associator satisfies a differential equation, which relates it to iterated Eisenstein integrals the derivations $\widetilde{\varepsilon}_{2k}$ of the last section. The results of this section are essentially due to Enriquez [31], however, our presentation slightly differs from [31].
Consider the differential one-form

$$\omega_{\mathrm{Eis}}^{\mathrm{geom}} := \sum_{k\geq 0} \underline{E}_{2k}(\tau) \cdot \widetilde{\varepsilon}_{2k}, \tag{5.29}$$

with values in the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$. Let $\mathcal{U}(\mathfrak{u}^{\mathrm{geom}})^\wedge$ be the completion (with respect to the lower central series) of the universal enveloping algebra with complex coefficients. Elements of $\mathcal{U}(\mathfrak{u}^{\mathrm{geom}})^\wedge$ are formal series with complex coefficients of the monomials

$$\widetilde{\varepsilon}_{\underline{2k}} := \widetilde{\varepsilon}_{2k_1} \circ \ldots \circ \widetilde{\varepsilon}_{2k_n} \in \mathcal{U}(\mathfrak{u}^{\mathrm{geom}}), \quad \underline{2k} = (2k_1, \ldots, 2k_n) \in (2\mathbb{Z}_{\geq 0})^n. \tag{5.30}$$

For a function

$$f : \mathbb{H} \to \mathcal{U}(\mathfrak{u}^{\mathrm{geom}})^\wedge, \tag{5.31}$$

we consider the differential equation

$$\mathrm{d}f = -\omega_{\mathrm{Eis}}^{\mathrm{geom}} \cdot f \tag{5.32}$$

As in Section 5.1.1, one can show that the limit

$$g(\tau) := \lim_{\rho\to i\infty} \exp\left[\int_\tau^\rho \omega_{\mathrm{Eis}}\right] \exp(\rho \cdot \mathrm{res}_\infty^{\mathrm{geom}}), \quad \mathrm{res}_\infty^{\mathrm{geom}} := \widetilde{\varepsilon}_0 + \sum_{k\geq 1} \frac{B_{2k}}{4k} \cdot \widetilde{\varepsilon}_{2k} \tag{5.33}$$

exists, and that (5.33) satisfies (5.32), with the boundary condition $g(\tau) \sim \exp(\tau \cdot \mathrm{res}_\infty^{\mathrm{geom}})$ as $\tau \to i\infty$. Since every $\widetilde{\varepsilon}_{2k}$ is a derivation of the free Lie algebra $\mathcal{L}$, it follows that $g(\tau)$ is an automorphism of $\mathbb{C}\langle\langle a, b\rangle\rangle$. Moreover, by virtue of Proposition A.2.3, we even have the following result.

Proposition 5.2.1. *There exists a derivation $r(\tau) \in \mathrm{Der}(\widehat{\mathcal{L}})$ of the completed Lie algebra $\widehat{\mathcal{L}}$, such that*

$$g(\tau) = \exp(r(\tau)). \tag{5.34}$$

In other words, $g(\tau)$ is group-like.

The series $g(\tau)$ can be written down explicitly using the iterated Eisenstein integrals $\mathcal{E}(\underline{2k}; \tau)$ (5.11) as

$$g(\tau) = \sum_{\underline{2k}} \mathcal{E}(\underline{2k}; \tau)\widetilde{\varepsilon}_{\underline{2k}}, \tag{5.35}$$

where the sum is over all multi-indices $\underline{2k} \in (2\mathbb{Z}_{\geq})^n$, for all $n \geq 0$.
The following theorem of Enriquez draws a connection between iterated Eisenstein integrals, the derivations $\widetilde{\varepsilon}_{2k}$ and elliptic multiple zeta values.

Theorem 5.2.2 (Enriquez)**.** *The generating series $\underline{A}(\tau)$ and $\underline{B}(\tau)$ of A-elliptic resp. B-elliptic multiple zeta values satisfy the differential equation*

$$\mathrm{d}h(\tau) = -\omega^{\mathrm{geom}}_{\mathrm{Eis}} \cdot h(\tau), \tag{5.36}$$

with boundary conditions

$$\underline{A}(\tau) \sim e^{\pi i t}\Phi(\widetilde{y}, t)e^{2\pi i \widetilde{y}}\Phi(\widetilde{y}, t)^{-1} \tag{5.37}$$

$$\underline{B}(\tau) \sim \Phi(-\widetilde{y} - t, t)e^{a}e^{\widetilde{y}\tau}\Phi(\widetilde{y}, t)^{-1}, \tag{5.38}$$

as $\tau \to i\infty$. Here, the variables $\widetilde{y}$ and t are given by

$$\widetilde{y} = -\frac{\mathrm{ad}(a)}{\exp(\mathrm{ad}(a)) - 1}(b), \quad t = -[a, b]. \tag{5.39}$$

We note that our conventions are slightly different from Enriquez's. More precisely, Enriquez uses variables x_0, x_1 and sets $t = -[x_0, x_1]$ and $\widetilde{y} = -\frac{\mathrm{ad}(x_0)}{e^{\mathrm{ad}(x_1)}-1}(x_1)$. On the other hand, we use variables $a := 2\pi i x_0$ and $b := (2\pi i)^{-1}x_1$, and set $t = -[a, b]$ and $\widetilde{y} = -\frac{\mathrm{ad}(a)}{e^{\mathrm{ad}(a)}-1}(b)$.

Proof: See [32], 5.2 (see also [31], Proposition 6.3). □

Solving this differential equation using Picard iteration, one obtains the following explicit formula for $\underline{A}(\tau)$ and $\underline{B}(\tau)$.

Corollary 5.2.3. *We have the equalities*

$$\underline{A}(\tau) = g(\tau)(\underline{A}_\infty), \quad \underline{B}(\tau) = g(\tau)(\underline{B}_\infty) \tag{5.40}$$

where $\underline{A}_\infty$ and $\underline{B}_\infty$ are given by

$$\underline{A}_\infty = e^{\pi i t}\Phi(\widetilde{y}, t)e^{2\pi i \widetilde{y}}\Phi(\widetilde{y}, t)^{-1} \tag{5.41}$$

$$\underline{B}_\infty = \Phi(-\widetilde{y} - t, t)e^{a}\Phi(\widetilde{y}, t)^{-1}. \tag{5.42}$$

Proof: It is enough to show that $g(\tau)(\underline{A}_\infty)$ and $g(\tau)(\underline{B}_\infty)$

(i) satisfy the differential equation 5.36, and

(ii) have the correct asymptotic behavior 5.37, 5.38.

i) follows from the definition of $g(\tau)$ and the discussion in Section A.2.2. For the asymptotic behavior, first note that $g(\tau) \sim \exp(\tau \cdot \mathrm{res}^{\mathrm{geom}}_\infty)$, where $\mathrm{res}^{\mathrm{geom}}_\infty = \widetilde{\varepsilon}_0 + \sum_{k\geq 1} \frac{B_{2k}}{4k}\widetilde{\varepsilon}_{2k}$. It is proved in [24] (proof of Proposition 4.9) that $\mathrm{res}^{\mathrm{geom}}_\infty$ annihilates both $\widetilde{y}$ and t. On the other hand, by Lemma 4.15 of [24] we have

$$\exp(\tau \cdot \mathrm{res}^{\mathrm{geom}}_\infty)(e^{a}) = e^{a}e^{\widetilde{y}\tau}. \tag{5.43}$$

From this, one deduces that

$$\exp(\tau \cdot \mathrm{res}_\infty^{\mathrm{geom}})(\underline{A}_\infty) = \underline{A}_\infty \tag{5.44}$$

$$\exp(\tau \cdot \mathrm{res}_\infty^{\mathrm{geom}})(\underline{B}_\infty) = \Phi(-\widetilde{y} - t, 2\pi i)e^a e^{\widetilde{y}\tau}\Phi(\widetilde{y}, t)^{-1} \tag{5.45}$$

(cf. [31], proof of Proposition 6.3), which proves (ii). □

5.3 The canonical embeddings

In this section, we employ the differential equation satisfied by the series $\underline{A}(\tau)$ and $\underline{B}(\tau)$ to deduce results about the structure of the $\mathbb{Q}$-algebras $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$ of A-elliptic resp. B-elliptic multiple zeta values.

In fact, we will replace $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$ by the slightly modified spaces

$$\overline{\mathcal{EZ}}^{\mathrm{A}} = \mathrm{Span}_{\mathbb{Q}}\{(2\pi i)^{-d(w)}\underline{A}(\tau)_w \mid w \in \langle x_0, x_1\rangle\} \tag{5.46}$$

$$\overline{\mathcal{EZ}}^{\mathrm{B}} = \mathrm{Span}_{\mathbb{Q}}\{(2\pi i)^{-d(w)}\underline{B}(\tau)_w \mid w \in \langle x_0, x_1\rangle\}, \tag{5.47}$$

where $d(w) = \deg_{x_0}(w) - \deg_{x_1}(w)$. This convention is chosen in particular to remove powers of $2\pi i$ from the denominators of $\underline{A}_\infty$ and $\underline{B}_\infty$. A similar proof as in Proposition 3.4.7, using that the series $\underline{A}(\tau)_w$ and $\underline{B}(\tau)_w$ are group-like, shows that $\overline{\mathcal{EZ}}^{\mathrm{A}}$ and $\overline{\mathcal{EZ}}^{\mathrm{B}}$ are in fact $\mathbb{Q}$-algebras.

A central role will be played by the universal enveloping algebra $U(\mathfrak{u}^{\mathrm{geom}})$ of the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$. Since $U(\mathfrak{u}^{\mathrm{geom}})$ is generated, as a (non-commutative) $\mathbb{Q}$-algebra, by the elements $\widetilde{\varepsilon}_{2k}$, we get a natural surjection

$$\begin{aligned} T(\mathbf{e}) &\to U(\mathfrak{u}^{\mathrm{geom}}) \\ \mathbf{e}_{2k} &\mapsto \widetilde{\varepsilon}_{2k}, \end{aligned} \tag{5.48}$$

which induces a dual embedding of $\mathbb{Q}$-algebras

$$U(\mathfrak{u}^{\mathrm{geom}})^\vee \hookrightarrow T(\mathbf{e})^\vee \cong \langle \mathcal{E}\rangle_{\mathbb{Q}}, \tag{5.49}$$

where the last isomorphism comes from Theorem 5.1.3.

There is an equivalent way of looking at the embedding (5.49). Namely, the element $g(\tau)$ of the previous section yields a morphism

$$\begin{aligned} U(\mathfrak{u}^{\mathrm{geom}})^\vee &\to \langle \mathcal{E}\rangle_{\mathbb{Q}} \\ \varepsilon_{\underline{2k}}^\vee &\mapsto \varepsilon_{\underline{2k}}^\vee(g(\tau))), \end{aligned} \tag{5.50}$$

which is clearly dual to the natural surjection (5.48), hence is precisely (5.49).

The following theorem gives a relation between $U(\mathfrak{u}^{\mathrm{geom}})^\vee$ and elliptic multiple zeta values.

Theorem 5.3.1. *We have canonical embeddings of $\mathbb{Q}$-algebras*

$$\iota_A : \overline{\mathcal{EZ}}^{\mathrm{A}} \hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^\vee \otimes \mathcal{Z}[2\pi i] \hookrightarrow T(\mathbf{e})^\vee \otimes \mathcal{Z}[2\pi i] \tag{5.51}$$

$$\iota_B : \overline{\mathcal{EZ}}^{\mathrm{B}} \hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^\vee \otimes \mathcal{Z} \hookrightarrow T(\mathbf{e})^\vee \otimes \mathcal{Z}, \tag{5.52}$$

where the algebras $\overline{\mathcal{EZ}}^{\mathrm{A}}, \overline{\mathcal{EZ}}^{\mathrm{B}}$ *have been defined in* (5.46) *and* (5.47).

Proof: The proof for $\overline{\mathcal{EZ}}^{\mathrm{A}}$ is almost identical to the one for $\overline{\mathcal{EZ}}^{\mathrm{B}}$. By definition, $\overline{\mathcal{EZ}}^{\mathrm{A}}$ is linearly spanned by the coefficients of $\underline{A}(\tau)$. On the other hand, by Corollary 5.2.3, we have $\underline{A}(\tau) = g(\tau)(\underline{A}_\infty)$. The coefficients of $\underline{A}_\infty$ are contained in $\mathcal{Z}[2\pi i]$, which follows immediately from the explicit formula (5.41). From the injectivity of the morphism (5.49) and the ensuing discussion concerning $g(\tau)$, we see that the coefficients of $g(\tau)$ (of elements $\varepsilon_{\underline{2k}}$) span a $\mathbb{Q}$-subalgebra of $\langle \mathcal{E} \rangle_{\mathbb{Q}}$, which is naturally isomorphic to $U(\mathfrak{u}^{\mathrm{geom}})^\vee$. It follows that the coefficients of $g(\tau)(\underline{A}_\infty)$, which are given by $\mathcal{Z}[2\pi i]$-linear combinations of iterated Eisenstein integrals span a $\mathbb{Q}$-subalgebra of $\langle \mathcal{E} \rangle_{\mathbb{Q}} \otimes \mathcal{Z}[2\pi i]$, which can be embedded into $U(\mathfrak{u}^{\mathrm{geom}})^\vee \otimes \mathcal{Z}[2\pi i]$. This gives the morphism

$$\iota_A : \overline{\mathcal{EZ}}^{\mathrm{A}} \hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^\vee \otimes \mathcal{Z}[2\pi i] \hookrightarrow \langle \mathcal{E} \rangle_{\mathbb{Q}} \otimes \mathcal{Z}[2\pi i]. \tag{5.53}$$

The construction of the morphism ι_B is identically the same, upon replacing $\underline{A}_\infty$ by $\underline{B}_\infty$, and using that $\underline{B}(\tau) = g(\tau)(\underline{B}_\infty)$ and that $\overline{\mathcal{EZ}}^{\mathrm{B}}$ is the $\mathbb{Q}$-algebra spanned by the coefficients of $\underline{B}(\tau)$ (the reason why we can replace $\mathcal{Z}[2\pi i]$ with $\mathcal{Z}$ in the definition of ι_B is that the coefficients of $\underline{B}_\infty$ are contained in $\mathcal{Z}$). □

Remark 5.3.2. (i) It is known that the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ is not freely generated by the derivations ε_{2k}, and that non-trivial relations between iterated Lie brackets of the ε_{2k} are closely related to cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ [61, 45]. As a consequence, $U(\mathfrak{u}^{\mathrm{geom}})^\vee$ is embedded into $T(\mathbf{e})^\vee$ as a proper subalgebra, hence Theorem 5.3.1 restricts the possible linear combinations of iterated Eisenstein integrals, which can possibly occur as elliptic multiple zeta values. In a slightly different way, this phenomenon was described also in [14].

(ii) The graded dual $U(\mathfrak{u}^{\mathrm{geom}})^\vee$ of the universal enveloping algebra of $\mathfrak{u}^{\mathrm{geom}}$ carries a natural coproduct Δ. Under the embedding (5.49), it corresponds to the deconcatenation coproduct on $T(\mathbf{e})^\vee$, given by

$$\Delta(\mathbf{e}^\vee_{2k_1} \dots \mathbf{e}^\vee_{2k_n}) = \sum_{i=0}^{n} \mathbf{e}^\vee_{2k_1} \dots \mathbf{e}^\vee_{2k_i} \otimes \mathbf{e}^\vee_{2k_{i+1}} \dots \mathbf{e}^\vee_{2k_n}. \tag{5.54}$$

The appearance of the deconcatenation coproduct in the context of elliptic multiple zeta values establishes a parallel between elliptic multiple zeta values and motivic multiple zeta values [16, 17, 40]. Indeed, the $\mathbb{Q}$-algebra $\mathcal{Z}^{\mathfrak{m}}/\langle\zeta^{\mathfrak{m}}(2)\rangle$ of motivic multiple zeta values (modulo the ideal generated by $\zeta^{\mathfrak{m}}(2)$) has a structure of Hopf algebra, whose coproduct is given by the Goncharov coproduct. It is known [16, 17] that under a suitable isomorphism $\phi : \mathcal{Z}^{\mathfrak{m}}/\langle\zeta^{\mathfrak{m}}(2)\rangle \cong \mathbb{Q}\langle f_3, f_5, f_7, \ldots\rangle$, the Goncharov coproduct corresponds to the deconcatenation coproduct on the shuffle algebra $\mathbb{Q}\langle f_3, f_5, f_7, \ldots\rangle$.

5.4 Length decomposition of elliptic multiple zeta values

We would like to understand better the images of the morphisms ι_A and ι_B, in particular, how the notion of length of elliptic multiple zeta values relates to the length of iterated Eisenstein integrals. The following proposition shows that the length n of an elliptic multiple zeta value gives an upper bound for the length of iterated Eisenstein integrals that can appear in the decomposition, and that the bound is in fact $n-1$ for A-elliptic multiple zeta values. Since iterated Eisenstein integrals of different length are linearly independent, the length expansion gives an useful way of decomposing an elliptic multiple zeta value into separate objects, which can be studied individually. This study will be the subject of later subsections.

Proposition 5.4.1. *For every $n \geq 0$, the $\mathbb{Q}$-algebra homomorphisms ι_A and ι_B restrict to $\mathbb{Q}$-linear maps*

$$\begin{aligned} \mathcal{L}_n(\overline{\mathcal{EZ}}^{\mathrm{A}}) &\hookrightarrow \bigoplus_{k\geq 0} \mathcal{L}_k(\langle\mathcal{E}\rangle_{\mathbb{Q}}) \otimes \mathcal{Z}[2\pi i] \\ \underline{A}(\tau)_w &\mapsto ((g_k(\tau)\underline{A}_\infty)_w)_{k=0,\ldots,n-1}, \end{aligned} \tag{5.55}$$

and

$$\begin{aligned} \mathcal{L}_n(\overline{\mathcal{EZ}}^{\mathrm{B}}) &\hookrightarrow \bigoplus_{k\geq 0} \mathcal{L}_k(\langle\mathcal{E}\rangle_{\mathbb{Q}}) \otimes \mathcal{Z} \\ \underline{B}(\tau)_w &\mapsto ((g_k(\tau)\underline{B}_\infty)_w)_{k=0,\ldots,n}. \end{aligned} \tag{5.56}$$

Furthermore, $(g_n(\tau)\underline{A}_\infty)_w = 0$.

We will call the expansions

$$((g_k(\tau)\underline{A}_\infty)_w)_{k=0,\ldots,n-1}, \quad ((g_k(\tau)\underline{B}_\infty)_w)_{k=0,\ldots,n} \tag{5.57}$$

the *length decomposition* of $\underline{A}(\tau)_w$ and $\underline{B}(\tau)_w$ respectively.

Proof: Let $w \in \langle a, b\rangle$ be a word with $\deg_b(w) = n$, so that[2]

$$\underline{A}(\tau)_w \in \mathcal{L}_n(\overline{\mathcal{EZ}}^{\mathrm{A}}), \quad \underline{B}(\tau)_w \in \mathcal{L}_n(\overline{\mathcal{EZ}}^{\mathrm{B}}). \tag{5.58}$$

[2] Previously, the length was defined for a word w on the alphabet $\langle x_0, x_1\rangle$ to be the x_1-degree of w. However, since $b = x_1$, there is no possible confusion.

By definition, ε_{2k} is homogeneous of b-degree equal to one, i.e. application of ε_{2k} increases the number of occurrences of the letter b by one. Together with Theorem 5.2.2, this implies that

$$\underline{A}(\tau)_w = \sum_{k=0}^{n} (g_k(\tau)\underline{A}_\infty)_w, \quad \underline{B}(\tau)_w = \sum_{k=0}^{n} (g_k(\tau)\underline{B}_\infty)_w. \tag{5.59}$$

Now $(g_k(\tau)\underline{A}_\infty)_w, (g_k(\tau)\underline{B}_\infty)_w \in \mathcal{L}_k\mathcal{Z}[2\pi i]\langle\mathcal{G}\rangle$, and since by Theorem 5.1.3 we have the direct sum decomposition

$$\mathcal{Z}[(2\pi i)^{-1}]\langle\mathcal{G}\rangle = \bigoplus_{k\geq 0} \mathcal{L}_k(\mathcal{Z}[(2\pi i)^{-1}])\langle\mathcal{G}\rangle, \tag{5.60}$$

it follows that (5.55) and (5.56) are injections.
It remains to show that $(g_n(\tau)\underline{A}_\infty)_w = 0$. For this, recall that

$$\underline{A}_\infty = e^{\pi i t}\Phi(\widetilde{y}, t)e^{2\pi i \widetilde{y}}\Phi(\widetilde{y}, t)^{-1}. \tag{5.61}$$

Both $t = -[a, b]$ and $\widetilde{y} = -\frac{\mathrm{ad}(a)}{\exp(\mathrm{ad}(a))-1}(b)$ are homogeneous of b-degree equal to one. Therefore, every term in $\underline{A}_\infty$ is of b-degree ≥ 1, and since ε_{2k} is homogeneous of degree one in b, we have for all $k_1, \ldots, k_n \geq 0$ that $(\varepsilon_{2k_1} \circ \ldots \circ \varepsilon_{2k_n})(\underline{A}_\infty)$ consists only of words of b-degrees $\geq n + 1$. In particular, it follows that

$$(g_n(\tau)\underline{A}_\infty)_w = 0. \tag{5.62}$$

□

5.4.1 The lowest length component

In this section, we study the coefficients of the series

$$\underline{A}_\infty = e^{\pi i t}\Phi(\widetilde{y}, t)e^{2\pi i \widetilde{y}}\Phi(\widetilde{y}, t)^{-1} \tag{5.63}$$

$$\underline{B}_\infty = \Phi(-\widetilde{y} - t, t)e^{a}\Phi(\widetilde{y}, t)^{-1}, \tag{5.64}$$

Since the coefficients of the Drinfeld associator are given by $\mathbb{Q}$-linear combinations of multiple zeta values, every coefficient of $\underline{A}_\infty$ is contained in $\mathcal{Z}[2\pi i]$ and that every coefficient of $\underline{B}_\infty$ is contained in $\mathcal{Z}$. We denote by

$$\alpha : \mathbb{Q}\langle a, b\rangle \to \mathcal{Z}[2\pi i] \tag{5.65}$$

$$\beta : \mathbb{Q}\langle a, b\rangle \to \mathcal{Z} \tag{5.66}$$

the $\mathbb{Q}$-linear morphisms, which map a word w onto its coefficient in $\underline{A}_\infty$, resp. its coefficient in $\underline{B}_\infty$.

Theorem 5.4.2. (i) *The image of the morphism* α *equals* $\mathbb{Q} + 2\pi i\mathcal{Z}[2\pi i]$.

(ii) *The morphism β is surjective.*

We need two lemmas first.

Lemma 5.4.3. *Let $\underline{k} = (k_1, \dots, k_n)$ be a multi-index with $k_i \geq 1$ and $k_1 \geq 2$, and consider the word*

$$w_{\underline{k}} = b^{k_1} a \dots b^{k_n} a. \tag{5.67}$$

Assume that some word $w' \in \langle b, t \rangle$ contains $w_{\underline{k}}$ with a non-trivial coefficient (this means that in the expansion of w' in a, b, the word $w_{\underline{k}}$ appears as a non-trivial summand). Then

$$w' = x_1^{k'_1} t \dots x_1^{k'_n} t \tag{5.68}$$

for some $k'_i \geq 1$, $k'_1 \geq 2$, with $(k'_1, \dots, k'_n) \preceq (k_1, \dots, k_n)$ where $\preceq$ denotes the reverse lexicographical ordering.

Proof: Since $w_{\underline{k}}$ ends with a and begins with two successive b's, it is clear that w' must begin with b and end with t. Also, every a in $w_{\underline{k}}$ must team up with either its left or its right neighboring b to arise from a commutator $[a, b]$. There are four possibilities for the two successive a's in the i-th and $(i+1)$-th position to team up with neighboring b's, namely

$$\text{left/left} \rightsquigarrow k'_{i+1} = k_{i+1} \tag{5.69}$$
$$\text{left/right} \rightsquigarrow k'_{i+1} = k_{i+1} + 1 \tag{5.70}$$
$$\text{right/left} \rightsquigarrow k'_{i+1} = k_{i+1} - 1 \tag{5.71}$$
$$\text{right/right} \rightsquigarrow k'_{i+1} = k_{i+1}. \tag{5.72}$$

Since $w_{\underline{k}}$ ends with a, the last a in the n-th position must team up with its left neighbor b, which gives $k'_n = k_n$ or $k'_n = k_n - 1$. Continuing in this way, we see that a word $b^{k'_1} t \dots b^{k'_n} t$ can contain the word $w_{\underline{k}}$, only if $(k'_1, \dots, k'_n) \preceq (k_1, \dots, k_n)$ in the reverse lexicographical ordering $\preceq$. □

The second result we need is the following

Lemma 5.4.4. *The coefficient of $w_{\underline{k}}$ in the series $\Phi(b, t)$ equals*

$$(-1)^n \zeta(k_1, \dots, k_n) + \sum_{\underline{k}' \prec \underline{k}} \lambda_{\underline{k}'} \zeta(\underline{k}'), \tag{5.73}$$

where $\lambda_{\underline{k}'} \in \mathbb{Q}$. Likewise, the coefficient of $w_{\underline{k}}$ in the series $\Phi(b, t)^{-1}$ equals

$$(-1)^{n+1} \zeta(k_1, \dots, k_n) + \sum_{\underline{k}' \prec \underline{k}} \lambda_{\underline{k}'} \zeta(\underline{k}') \mod (\mathcal{Z}^2)_k, \tag{5.74}$$

where $\lambda_{\underline{k}} \in \mathbb{Q}$ and $(\mathcal{Z}^2)_k \subset \mathcal{Z}_k$ denotes the vector subspace of non-trivial products of weight k of multiple zeta values.

Proof: First, it is clear that

$$(\Phi(b,t))_{w_{\underline{k}}} = \sum_{w'} (\Phi(b,t))_{w'}, \tag{5.75}$$

where the sum is over all words $w' \in \langle b,t \rangle$, which contain $w_{\underline{k}}$ as a summand (via the expansion $t = ba - ab$). By Lemma 5.4.3, it thus follows that

$$(\Phi(b,t))_{w_{\underline{k}}} = \sum_{\underline{k}' \preceq \underline{k}} (\Phi(b,t))_{w'_{\underline{k}'}}, \tag{5.76}$$

for multi-indices $\underline{k}' = (k'_1, \dots, k'_n)$. We have seen in Chapter 1.3 that the coefficient of $w'_{\underline{k}'}$ in $\Phi(b,t)$ is equal to $(-1)^n \zeta(k'_1, \dots, k'_n)$, thus (5.73) follows. A similar argument shows that

$$(\Phi(b,t)^{-1})_{w_{\underline{k}}} = \sum_{\underline{k}' \preceq \underline{k}} (\Phi(b,t)^{-1})_{w'_{\underline{k}'}}. \tag{5.77}$$

We have

$$\Phi(b,t)^{-1}_{w'_{\underline{k}'}} \equiv -(-1)^n \zeta(k'_1, \dots, k'_n) = (-1)^{n+1} \zeta(k'_1, \dots, k'_n) \mod (\mathcal{Z}^2)_k, \tag{5.78}$$

which can be seen by comparing the coefficients of $w'_{\underline{k}'}$ on both sides of the equation $\Phi(b,t)\Phi(b,t)^{-1} = 1$, and (5.74) follows. □

Proof: (of Theorem 5.4.2) We first prove (i), i.e. that the image of α equals $\mathbb{Q} \oplus 2\pi i \mathcal{Z}$. Given a word $w \in \langle a,b \rangle$, we have by definition

$$\alpha(w) = (\underline{A}_\infty)_w = (e^{\pi i t} \Phi(\widetilde{y}, t) e^{2\pi i \widetilde{y}} \Phi(\widetilde{y}, t)^{-1})_w, \tag{5.79}$$

where $\widetilde{y} = -\frac{\mathrm{ad}(a)}{e^{\mathrm{ad}(a)}-1}(b) = -\sum_{n \geq 0} \frac{B_n}{n!} \mathrm{ad}^n(a)(b)$ and $t = -\mathrm{ad}(a)(b)$. It is easy to see that $\alpha(1_w) = 1$ (where 1_w denotes the empty word) and that $\alpha(b^n) = \frac{(-2\pi i)^n}{n!}$. Therefore, it remains to prove that for every multiple zeta value $\zeta(k_1, \dots, k_n)$, the element $2\pi i \zeta(k_1, \dots, k_n)$ is in the image of the morphism α.
To this end, for a multi-index $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 1}$ of weight $k = k_1 + \dots + k_n$ and depth n, we set $w_{\underline{k}} = b \cdot b^{k_n} a \dots b^{k_1} a$. For integers $m, n \geq 0$, let

$$\mathcal{Z}^{\leq n-1}_{\leq m} := \sum_{0 \leq k \leq m} \mathcal{D}_{n-1}(\mathcal{Z}_k) + \sum_{0 \leq k \leq m} \mathcal{D}_n(\mathcal{Z}_k) \cap \mathcal{Z}^2_{>0} \tag{5.80}$$

be the $\mathbb{Q}$-span of the multiple zeta values of weight at most m and depth at most $n-1$ and of all products of multiple zeta values, such that the product has depth at most n and weight at most k. We claim that

$$(\underline{A}_\infty)_w = -2\pi i (-1)^{k_1 + \dots + k_n - n} \zeta(k_1, \dots, k_n) + z_{k,n}, \tag{5.81}$$

where $z_{k,n} \in \mathcal{Z}^{\leq n-1}_{\leq k}$. By a straightforward double induction on weight and depth, we see that this claim implies (i).

To prove the claim, first note that

$$(\underline{A}_\infty)_{w_{\underline{k}}} = \sum_{w_{\underline{k}}=pqrs} (e^{\pi it})_p (\Phi(\widetilde{y},t))_q (e^{2\pi i\widetilde{y}})_r (\Phi(\widetilde{y},t)^{-1})_s. \tag{5.82}$$

No word occurring in the series $e^{\pi it}$ can have two consecutive b's. However, by construction $w_{\underline{k}}$ begins with two consecutive b's, thus the $e^{\pi it}$-term in (5.82) does not contribute to the sum, and we see that

$$\sum_{w_{\underline{k}}=pqrs} (e^{\pi it})_p (\Phi(\widetilde{y},t))_q (e^{2\pi i\widetilde{y}})_r (\Phi(\widetilde{y},t)^{-1})_s = \sum_{w_{\underline{k}}=qrs} (\Phi(\widetilde{y},t))_q (e^{2\pi i\widetilde{y}})_r (\Phi(\widetilde{y},t)^{-1})_s. \tag{5.83}$$

Now write

$$\begin{aligned} \underline{A}_\infty &= \sum_{w_{\underline{k}}=qrs} (\Phi(\widetilde{y},t))_q (e^{2\pi i\widetilde{y}})_r (\Phi(\widetilde{y},t)^{-1})_s \\ &= \sum_{w_{\underline{k}}=qs} (\Phi(\widetilde{y},t))_q (\Phi(\widetilde{y},t)^{-1})_s \\ &+ \sum_{w_{\underline{k}}=qr,\, r\neq 1} (\Phi(\widetilde{y},t))_q (e^{2\pi i\widetilde{y}})_r \\ &+ \sum_{w_{\underline{k}}=rs,\, r\neq 1} (e^{2\pi i\widetilde{y}})_r (\Phi(\widetilde{y},t)^{-1})_s \\ &+ \sum_{w_{\underline{k}}=qrs,\, q,r,s\neq 1} (\Phi(\widetilde{y},t))_q (e^{2\pi i\widetilde{y}})_r (\Phi(\widetilde{y},t)^{-1})_s. \end{aligned} \tag{5.84}$$

Since $w_{\underline{k}}$ is not the trivial word, the second line vanishes, and the last line is obviously contained in $Z_{k,n}$. For the third line, note that for every factorization $w_{\underline{k}} = bc$ with c non-trivial the word b has at most $n-1$ occurrences of the letter a, and thus $(\Phi(\widetilde{y},t))_b$ has depth at most $n-1$. Consequently, also the third line is contained in $\mathcal{Z}_{\leq k}^{\leq n-1}$. Finally, the fourth line equals

$$(-1)^{k_1+\ldots+k_n+1} 2\pi i\zeta(k_1,\ldots,k_n) + \sum_{\underline{k}'\prec\underline{k}} \lambda_{\underline{k}'}\zeta(\underline{k}') + \underbrace{\sum_{w_{\underline{k}}=rs,\, r\neq 1,q} (e^{2\pi i\widetilde{y}})_r (\Phi(\widetilde{y},t)^{-1})_s}_{\in Z_{k,n}}, \tag{5.85}$$

with $\lambda_{\underline{k}'} \in \mathbb{Q}$, by Lemma 5.4.4, since $\widetilde{y} = -b + $ higher terms. All in all, this proves the claim, and hence (i).

We now prove ii), i.e. the surjectivity of β. By definition, we have

$$\beta(w) = (\underline{B}_\infty)_w = (\Phi(-\widetilde{y}-t,t)e^a\Phi(\widetilde{y},t)^{-1})_w \tag{5.86}$$

For a multi-index $\underline{k} \in \mathbb{Z}_{\geq 1}^n$ of weight k and depth n, set $w_{\underline{k}} = b^{k_n}a\ldots b^{k_1}a\cdot a$. We claim that

$$(\underline{B}_\infty)_{w_{\underline{k}}} = \zeta(k_1,\ldots,k_n) + z_{\underline{k}}, \tag{5.87}$$

where $z_{\underline{k}} \in Z_{m,n}$, and the surjectivity of $\underline{B}_\infty$ follows from this. In order to prove the claim, note that

$$\begin{aligned}(\underline{B}_\infty)_{w_{\underline{k}}} = &\sum_{w_{\underline{k}}=pqr} (\Phi(-\widetilde{y}-t,t))_p(e^a)_q(\Phi(\widetilde{y},t)^{-1})_r \\ &+ \sum_{w_{\underline{k}}=pqr,\, p,q,r\neq 1} (\Phi(-\widetilde{y}-t,t))_p(e^a)_q(\Phi(\widetilde{y},t)^{-1})_r \\ &+ \sum_{w_{\underline{k}}=qr,\, q,r\neq 1} (e^a)_q(\Phi(\widetilde{y},t)^{-1})_r \\ &+ \sum_{w_{\underline{k}}=pq,\, p,q\neq 1} (\Phi(-\widetilde{y}-t,t))_p(e^a)_q \\ &+ \sum_{w_{\underline{k}}=pr,\, p,r\neq 1} (\Phi(-\widetilde{y}-t,t))_p(\Phi(\widetilde{y},t)^{-1})_r \\ &+ (\Phi(-\widetilde{y}-t,t))_{w_{\underline{k}}} + (e^a)_{w_{\underline{k}}} + (\Phi(\widetilde{y},t)^{-1})_{w_{\underline{k}}}.\end{aligned} \tag{5.88}$$

Clearly the second and fifth line are contained in $Z_{m,n}$ and the third line vanishes, since $w_{\underline{k}}$ starts with b, but the series e^a does not contain any b-term. Again by Lemma 5.4.4, the fourth line is equal to

$$\zeta(k_1,\dots,k_n) + \sum_{\underline{k}'\prec\underline{k}} \lambda_{\underline{k}'}\zeta(\underline{k}') + \underbrace{(\Phi(-\widetilde{y}-t,t))_{b^{k_n}a\dots b^{k_n}}(e^a)_{a^2}}_{\in Z_{m,n}}, \tag{5.89}$$

while the last line is again contained in $Z_{m,n}$. The last fact can be seen as follows: since $w_{\underline{k}}$ ends with two a's, the only words in $\widetilde{y}$ and t, which can give rise to $w_{\underline{k}}$ must end with $\widetilde{y}$. More precisely, these words are of the form $v\widetilde{y}$, where v is a word with at most $n-1$ occurrences of the word t, hence the depth of the multiple zeta values occurring can be at most $n-1$, and it follows that both $(\Phi(-\widetilde{y}-t,t))_{w_{\underline{k}}}$ and $(\Phi(\widetilde{y},t)^{-1})_{w_{\underline{k}}}$ are contained in $Z_{k,n}$. Finally, it is clear that $(e^a)_{w_{\underline{k}}} = 0$, since $w_{\underline{k}}$ contains the letter b. This ends the proof of Theorem 5.4.2. □

5.4.2 The highest length component

In this section, we study the highest length component of the length decomposition. We first make precise what we mean by "highest length component".

Definition 5.4.5. Let $w \in \langle a,b\rangle$ be a word of length n. The *highest length component* of $\underline{A}(\tau)_w$, resp. of $\underline{B}(\tau)_w$ is

$$(g_{n-1}(\tau)\underline{A}_\infty)_w, \quad \text{resp. } (g_n(\tau)\underline{B}_\infty)_w. \tag{5.90}$$

The following proposition gives an alternative characterization of the highest length component.

Proposition 5.4.6. *Let w be a word of length n. We have the equalities*

$$(g_{n-1}(\tau)\underline{A}_\infty)_w = g(\tau)(\pi i t + 2\pi i\widetilde{y})_w, \quad (g_n(\tau)\underline{B}_\infty)_w = g(\tau)(e^a)_w, \tag{5.91}$$

Proof: It follows from the definitions of $\underline{A}_\infty$ and $\underline{B}_\infty$ that

$$\underline{A}_\infty = \pi i t + 2\pi i\widetilde{y} + \text{ terms of } b\text{-degree} \geq 2 \tag{5.92}$$

$$\underline{B}_\infty = e^a + \text{ terms of } b\text{-degree} \geq 1. \tag{5.93}$$

Since every $\widetilde{\varepsilon}_{2k}$ annihilates t and increases the b-degree (i.e. the length) by one, the proposition follows. □

Definition 5.4.7. Define two $\mathbb{Q}$-vector spaces $\overline{\mathcal{EZ}}^{\text{A}-\text{geom}}$ and $\overline{\mathcal{EZ}}^{\text{B}-\text{geom}}$ by

$$\overline{\mathcal{EZ}}^{\text{A}-\text{geom}} = \text{Span}_\mathbb{Q}\{(g(\tau)(e^{\frac{t}{2}+\tilde{y}}))_w \,|\, w \in \langle a, b\rangle\} \tag{5.94}$$

$$\overline{\mathcal{EZ}}^{\text{B}-\text{geom}} = \text{Span}_\mathbb{Q}\{(g(\tau)(e^{a}))_w \,|\, w \in \langle a, b\rangle\}. \tag{5.95}$$

Proposition 5.4.8. (i) *Both $\overline{\mathcal{EZ}}^{\text{A}-\text{geom}}$ and $\overline{\mathcal{EZ}}^{\text{B}-\text{geom}}$ are $\mathbb{Q}$-algebras.*

(ii) *The algebra $\overline{\mathcal{EZ}}^{\text{A}-\text{geom}}$ is the algebra generated by the highest length components of A-elliptic multiple zeta values[3]. The algebra $\overline{\mathcal{EZ}}^{\text{B}-\text{geom}}$ is the $\mathbb{Q}$-vector space linearly spanned by the highest length components of B-elliptic multiple zeta values.*

Proof: (i) Since $g(\tau) = \exp(r(\tau))$, where $r(\tau)$ is a formal series of derivations of $\mathcal{L}$, it follows that $g(\tau)$ is an automorphism of $\exp(\widehat{\mathcal{L}})$. By Proposition A.1.9, it follows that $g(\tau)(e^{\frac{t}{2}+\tilde{y}})$ and $g(\tau)(e^a)$ are both group-like, therefore the $\mathbb{Q}$-span of their coefficients each generate a $\mathbb{Q}$-algebra by Corollary A.1.8

(ii) By Proposition 5.4.6, we know that the $\mathbb{Q}$-vector space spanned by the highest length components of A-elliptic multiple zeta values is spanned by the coefficients of the series $g(\tau)(t/2+\tilde{y})$ (recall that we removed the $2\pi i$-prefactors). Since $g(\tau)$ is an automorphism, it follows that

$$\exp(g(\tau)(t/2+\tilde{y})) = g(\tau)(e^{\frac{t}{2}+\tilde{y}}), \quad \log(g(\tau)(e^{\frac{t}{2}+\tilde{y}})) = g(\tau)(t/2+\tilde{y}), \tag{5.96}$$

and therefore the coefficients of $g(\tau)(t/2+\tilde{y})$ are algebraic combinations of the coefficients of $g(\tau)(e^{\frac{t}{2}+\tilde{y}})$ and vice-versa. In particular, the coefficients of the two series generate the same $\mathbb{Q}$-algebra, namely $\overline{\mathcal{EZ}}^{\text{A}-\text{geom}}$.

Finally, the statement that $\overline{\mathcal{EZ}}^{\text{B}-\text{geom}}$ is linearly spanned by the highest length components of B-elliptic multiple zeta values follows directly from Proposition 5.4.6.

□

[3] We have chosen to remove the slightly distracting $2\pi i$-prefactors

Next, we determine the structure of the $\mathbb{Q}$-algebra $\overline{\mathcal{EZ}}^{\mathrm{B-geom}}$. To this end, we need the following

Proposition 5.4.9. *We have*

$$\mathrm{Alg}_{\mathbb{Q}}\{r(\tau)(a)\big|_w \mid w \in \langle a,b\rangle\} = \mathrm{Alg}_{\mathbb{Q}}\{g(\tau)(a)\big|_w \mid w \in \langle a,b\rangle\}, \tag{5.97}$$

where for a subset $S \subset \mathcal{O}(\mathbb{H})$, $\mathrm{Alg}_{\mathbb{Q}} S$ denotes the $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathbb{H})$ generated by S.

Proof: Recall that $g(\tau) = \sum_{n\geq 0}\sum_{k_1,\dots,k_n\geq 0}\mathcal{G}(2k_1,\dots,2k_n;\tau)\varepsilon_{2k_1}\circ\dots\circ\varepsilon_{2k_n}$, and $r(\tau) = \log(g(\tau))$. Since the derivation algebra $\mathfrak{u}$ is graded for the (commutator) length, we can expand $r(\tau)$ in a homogeneous (for the length) basis $\mathcal{B}$ of $\mathfrak{u}$

$$r(\tau) = \sum_{f_i\in\mathcal{B}} c_{f_i} f_i, \tag{5.98}$$

where the f_i are commutators of the generators ε_{2k} of $\mathfrak{u}$, which are homogeneous for the length .

Now, for $n \geq 0$, let

$$A_r^n := \mathrm{Alg}_{\mathbb{Q}}^n\{r(\tau)(a)\big|_w \mid w \in \langle a,b\rangle,\, \ell(w)\leq n\}, \tag{5.99}$$

$$A_g^n := \mathrm{Alg}_{\mathbb{Q}}^n\{g(\tau)(a)\big|_w \mid w \in \langle a,b\rangle,\, \ell(w)\leq n\}, \tag{5.100}$$

where $\ell(w)$ denotes the length (number of occurrences of b) of the word w. It is clearly enough to prove that $A_r^n = A_g^n$ for all $n \geq 0$.

To this end, we use induction on $n \geq 0$. For $n = 0$, the statement is clear: the only words $w \in \langle a,b\rangle$ with $\ell(w) = 0$ are a and the empty word. Since $r(\tau)(a)$ contains neither, A_r^0 is the $\mathbb{Q}$-algebra generated by the empty set, which is $\mathbb{Q}$ (recall that the empty product is $1 \in \mathbb{Q}$ by definition). On the other hand,

$$g(\tau)(a) = \sum_{k=0}^{\infty}\frac{1}{k!}r(\tau)^k(a) = a + r(\tau)(a) + \frac{1}{2}r(\tau)^2(a) + \dots, \tag{5.101}$$

and since $g(\tau)(a)\big|_a = 1$, we have $\mathrm{Alg}_g^0 = \mathbb{Q}$ as well.

Now assume that for some $n \geq 1$, we have $\mathrm{Alg}_r^{n-1} = \mathrm{Alg}_g^{n-1}$, and let w be a word of length n. From (5.101), we see that (using that $a\big|_w = 0$, since $\ell(w) \geq 1$)

$$g(\tau)(a)\big|_w = r(\tau)(a)\big|_w + \sum_{k=2}^{\infty}\frac{1}{k!}r(\tau)^k(a)\big|_w. \tag{5.102}$$

We claim that $r(\tau)^k(a)\big|_w \in \mathrm{Alg}_r^{n-1}$ for every $k \geq 2$. In order to show the claim we proceed as follows. First, it is clear that

$$r(\tau)^k(a)\big|_w = \sum_{f_{i_1},\dots,f_{i_k}\in\mathcal{B}}\Big[c_{f_{i_1}}\cdot\ldots\cdot c_{f_{i_k}}(f_{i_1}\circ\dots\circ f_{i_k})(a)\Big]\big|_w. \tag{5.103}$$

Letting $c\ell(f_i)$ be the commutator length of f_i, we have $c\ell(f_i) = \ell(f_i(a))$, since every derivation ε_{2k} is homogeneous of length one, i.e. increases the number of b's by one (thus a commutator of ε_{2k}'s increases the number of b's by its commutator length). Using this, we can make (5.103) more precise:

$$r(\tau)^k(a)\big|_w = \sum_{c\ell(f_{i_1})+\dots c\ell(f_{i_k})=n} \Big[c_{f_{i_1}} \cdot \ldots \cdot c_{f_{i_k}} (f_{i_1} \circ \ldots \circ f_{i_k})(a)\Big]\big|_w. \tag{5.104}$$

Since $k \geq 2$, this means in particular that every f_{i_j} occurring in the above sum has commutator length strictly smaller than n. From this, we infer that

$$c_{f_{i_j}} \in \mathrm{Span}_{\mathbb{Q}}\{r(\tau)\big|_f \mid f \in \mathcal{B},\, c\ell(f) < n\}, \tag{5.105}$$

for every $c_{f_{i_j}}$ occurring in (5.104). However, since the map $\mathfrak{u} \to \mathrm{Lie}(a,b)$, mapping $f \mapsto f(a)$ is injective, and since $c\ell(f) = \ell(f(a))$, we have

$$\mathrm{Span}_{\mathbb{Q}}\{r(\tau)\big|_f \mid f \in \mathcal{B},\, c\ell(f) < n\} = \mathrm{Span}_{\mathbb{Q}}\{r(\tau)(a)\big|_w \mid w \in \langle a,b\rangle,\, \ell(w) < n\}. \tag{5.106}$$

The vector space on the right is contained in Alg_r^{n-1} by definition. In particular, every c_{f_i} occurring in (5.104) is in Alg_r^{n-1}, therefore $r(\tau)^k(a)\big|_w \in \mathrm{Alg}_r^{n-1}$.
Now that we have shown that $r(\tau)^k(a)\big|_w \in \mathrm{Alg}_r^{n-1}$, for every $k \geq 2$, we get from (5.102) and from our induction hypothesis

$$g(\tau)(a)\big|_w - r(\tau)(a)\big|_w \in \mathrm{Alg}_r^{n-1} = \mathrm{Alg}_g^{n-1}. \tag{5.107}$$

Therefore, it follows that $g(\tau)(a)\big|_w \in \mathrm{Alg}_r^n$, as well as $r(\tau)(a)\big|_w \in \mathrm{Alg}_g^n$, and the proposition is proved. □

Theorem 5.4.10. *There is a natural isomorphism of $\mathbb{Q}$-algebras.*

$$\overline{\mathcal{EZ}}^{\mathrm{B-geom}} \cong U(\mathfrak{u}^{\mathrm{geom}})^\vee. \tag{5.108}$$

Proof: Let $\mathfrak{u}_a^{\mathrm{geom}} \subset \mathcal{L}$ be the image of the evaluation map

$$\begin{aligned} \psi_a : \mathfrak{u}^{\mathrm{geom}} &\to \mathcal{L} \\ \delta &\mapsto \delta(a). \end{aligned} \tag{5.109}$$

Since ψ_a is injective (cf. Corollary 5.25), it induces an isomorphism of $\mathbb{Q}$-vector spaces $\mathfrak{u}^{\mathrm{geom}} \cong \mathfrak{u}_a^{\mathrm{geom}}$, which can be promoted to an isomorphism of Lie algebras, by transporting the Lie algebra structure on $\mathfrak{u}^{\mathrm{geom}}$ to $\mathfrak{u}_a^{\mathrm{geom}}$. In particular, we get an isomorphism of the corresponding universal enveloping algebras

$$U(\mathfrak{u}^{\mathrm{geom}}) \cong U(\mathfrak{u}_a^{\mathrm{geom}}), \tag{5.110}$$

which yields, when passing to the graded duals on both sides a natural isomorphism of $\mathbb{Q}$-algebras.

$$U(\mathfrak{u}_a^{\mathrm{geom}})^\vee \cong U(\mathfrak{u}^{\mathrm{geom}})^\vee. \tag{5.111}$$

We now claim that $U(\mathfrak{u}_a^{\mathrm{geom}})^\vee$ is naturally isomorphic to $\overline{\mathcal{EZ}}^{\mathrm{B-geom}}$ as a $\mathbb{Q}$-algebra. Indeed, by definition $\overline{\mathcal{EZ}}^{\mathrm{B-geom}}$ is spanned as a $\mathbb{Q}$-vector space by the coefficients of the series

$$g(\tau)(e^a) = \exp(g(\tau)(a)). \tag{5.112}$$

By Proposition 5.4.8, $\overline{\mathcal{EZ}}^{\mathrm{B-geom}}$ is also generated as a $\mathbb{Q}$-algebra by the coefficients of $r(\tau)(a)$. Evaluation at $r(\tau)(a)$ induces a $\mathbb{Q}$-linear morphism

$$r(\tau)(a) \in \mathrm{Hom}_{\mathbb{Q}}((\mathfrak{u}_a^{\mathrm{geom}})^\vee, \langle\mathcal{E}\rangle_{\mathbb{Q}}), \tag{5.113}$$

and the image of $r(\tau)(a)$ generates $\overline{\mathcal{EZ}}^{\mathrm{B-geom}}$ as a $\mathbb{Q}$-algebra. Equivalently, the element $r(\tau)(a)$ induces an isomorphism $(\overline{\mathcal{EZ}}^{\mathrm{B-geom}})^\vee \cong U(\mathfrak{u}_a^{\mathrm{geom}})$, and dualizing this isomorphism and composing with (5.111) finally yields the isomorphism

$$\overline{\mathcal{EZ}}^{\mathrm{B-geom}} \cong U(\mathfrak{u}^{\mathrm{geom}})^\vee. \tag{5.114}$$

□

5.4.3 A-elliptic multiple zeta values and the Fourier subspace

In this subsection, we will see that the analogous statement for A-elliptic multiple zeta values, namely that the highest length component is isomorphic to $U(\mathfrak{u}^{\mathrm{geom}})^\vee$, is not quite true.

Definition 5.4.11. The *Fourier subspace* $\langle\mathcal{E}\rangle_{\mathbb{Q}}^{\mathrm{Fou}} \subset \langle\mathcal{E}\rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$-linear subspace, defined by

$$\langle\mathcal{E}\rangle_{\mathbb{Q}}^{\mathrm{Fou}} := \mathrm{Span}_{\mathbb{Q}}\{\mathcal{E}^0(2k_1,\dots,2k_n;\tau) \,|\, n \geq 0,\ k_i \geq 0\}, \tag{5.115}$$

where $\mathcal{E}^0(2k_1,\dots,2k_{n-1},0;\tau) := 0$ and for $k_n \neq 0$, we set

$$\mathcal{E}^0(2k_1,\dots,2k_n;\tau) := \mathcal{E}(2k_1,\dots,2k_n;\tau) - \frac{B_{2k_n}}{4k_n}\mathcal{E}(2k_1,\dots,2k_{n-1},0;\tau). \tag{5.116}$$

We will denote by $T(\mathbf{e})^\vee_{\mathrm{Fou}}$ the subspace of $T(\mathbf{e})^\vee$, which is the image of $\langle\mathcal{E}\rangle_{\mathbb{Q}}^{\mathrm{Fou}}$ under the isomorphism $\langle\mathcal{E}\rangle_{\mathbb{Q}} \cong T(\mathbf{e})^\vee$ of Theorem 5.1.3.

Remark 5.4.12. By Remark 5.1.5, every iterated Eisenstein integral $\mathcal{E}(2k_1,\dots,2k_n;\tau)$ has an expansion in $q =$ and $\log(q) := 2\pi i\tau$ with rational coefficients. A $\mathbb{Q}$-linear combination of iterated Eisenstein integrals $\mathcal{E}(2k_1,\dots,2k_n;\tau)$ has a Fourier expansion $\sum_{n\geq 0} a_n q^n$

(i.e. the $\log(q)$ terms all vanish) if and only if it is contained in $\langle\mathcal{E}\rangle_{\mathbb{Q}}^{\mathrm{Fou}}$. This follows easily from the Fourier expansion $E_{2k}(\tau) = -\frac{B_{2k}}{4k} + O(q)$, valid for $k > 0$, which together with $E_0 := -1$ implies that $\mathcal{E}^0(2k_1, \dots, 2k_n; \tau) \in O(q)$, since the ideal $q \cdot \mathbb{Q}[[q]] \subset \mathbb{Q}[[q]]$ is closed under integration with respect to the measure $\frac{\mathrm{d}q}{q}$.

Theorem 5.4.13. *The embedding ι_A of Theorem 5.3.1 maps $\overline{\mathcal{EZ}}^{\mathrm{A}}$ into the Fourier subspace, more precisely*

$$\iota_A : \overline{\mathcal{EZ}}^{\mathrm{A}} \hookrightarrow U(\mathfrak{u}^{\mathrm{geom}})^{\vee}_{\mathrm{Fou}} \otimes \mathcal{Z}[2\pi i], \tag{5.117}$$

where $U(\mathfrak{u}^{\mathrm{geom}})^{\vee}_{\mathrm{Fou}} := U(\mathfrak{u}^{\mathrm{geom}})^{\vee} \cap T(\mathbf{e})^{\vee}_{\mathrm{Fou}}$.

Proof: We can rewrite $g(\tau)$ using the $\mathcal{E}^0(2k_1, \dots, 2k_n; \tau)$ as follows:

$$\begin{aligned} g(\tau) &= \sum_{\underline{2k}} \mathcal{E}(\underline{2k}; \tau) \widetilde{\varepsilon}_{\underline{2k}} \\ &= \sum_{\underline{2k},\, k_n \neq 0} \Big(\mathcal{E}^0(\underline{2k}; \tau) + \frac{B_{2k_n}}{4k_n} \mathcal{E}(2k_1, \dots, 2k_{n-1}, 0; \tau) \Big) \widetilde{\varepsilon}_{\underline{2k}} \\ &\qquad\qquad + \sum_{\underline{2k},\, k_n = 0} \mathcal{E}(2k_1, \dots, 2k_{n-1}, 0; \tau) \widetilde{\varepsilon}_{\underline{2k}} \circ \widetilde{\varepsilon}_0 \\ &= \sum_{\underline{2k}} \mathcal{E}^0(2k_1, \dots, 2k_n; \tau) \widetilde{\varepsilon}_{\underline{2k}} \\ &\quad + \sum_{\underline{2k}} \mathcal{E}(2k_1, \dots, 2k_{n-1}, 0; \tau) \widetilde{\varepsilon}_{\underline{2k}} \circ \underbrace{\Big(\widetilde{\varepsilon}_0 + \sum_{k_n \geq 1} \frac{B_{2k_n}}{4k_n} \widetilde{\varepsilon}_{2k} \Big)}_{\mathrm{res}_\infty^{\mathrm{geom}}}. \end{aligned} \tag{5.118}$$

Since $\mathrm{res}_\infty^{\mathrm{geom}}$ is a derivation that annihilates both $\widetilde{y}$ and t, it annihilates every word in $\widetilde{y}$ and t. It follows that $\mathrm{res}_\infty^{\mathrm{geom}}(\underline{A}_\infty) = 0$, since $\underline{A}_\infty$ is a formal power series in $\widetilde{y}$ and t. Thus,

$$\underline{A}(\tau) = g(\tau)(\underline{A}_\infty) = \Big(\sum_{\underline{2k}} \mathcal{E}^0(2k_1, \dots, 2k_n; \tau) \widetilde{\varepsilon}_{\underline{2k}} \Big) (\underline{A}_\infty), \tag{5.119}$$

and therefore every coefficient of $\underline{A}(\tau)$ is contained in $\langle\mathcal{E}\rangle_{\mathbb{Q}}^{\mathrm{Fou}} \otimes \mathcal{Z}[2\pi i]$. □

Remark 5.4.14. (i) Theorem 5.4.13 reproves a result of Enriquez, namely that every A-elliptic multiple zeta value has a Fourier expansion, whose coefficients are in $\mathcal{Z}[2\pi i]$ (cf. [32], Proposition 5.2).

(ii) It follows from Theorem 5.4.13 that the $\mathbb{Q}$-algebra $\overline{\mathcal{EZ}}^{\mathrm{A-geom}}$ is a $\mathbb{Q}$-subalgebra of $U(\mathfrak{u}^{\mathrm{geom}})^{\vee}_{\mathrm{Fou}}$. In analogy with Theorem 5.4.10, it is tempting to conjecture that

$$\overline{\mathcal{EZ}}^{\mathrm{A-geom}} \cong U(\mathfrak{u}^{\mathrm{geom}})^{\vee}_{\mathrm{Fou}}, \tag{5.120}$$

but the author doesn't know how to prove this.

Appendix A

Some background

In this part of the appendix, we collect some well-known facts from the theory of Lie algebras, as well as from the theory of iterated integrals. Results from this section are sometimes used implicitly in the arguments in the main text.

A.1 Lie algebras

In this section, k is an arbitrary commutative ring. We follow [72].

A.1.1 General definitions

Definition A.1.1. (i) A *Lie algebra* $\mathfrak{g}$ over k is a k-module together with a linear map

$$[\cdot,\cdot] : \mathfrak{g} \otimes_k \mathfrak{g} \to k \tag{A.1}$$

satisfying *antisymmetry*

$$[x,x] = 0, \quad \forall x \in \mathfrak{g} \tag{A.2}$$

as well as the *Jacobi identity*

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0, \quad \forall x,y,z \in \mathfrak{g}. \tag{A.3}$$

If the bracket $[\cdot,\cdot]$ is the zero map, $\mathfrak{g}$ is called *abelian*. Moreover, a Lie algebra $\mathfrak{g}$ is called *graded*, if there exists a decomposition

$$\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}_n \tag{A.4}$$

into k-submodules $\mathfrak{g}_n$, such that $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$ for all $k, l \geq 0$.

(ii) A *derivation* D on a Lie algebra $\mathfrak{g}$ is a k-linear map

$$D : \mathfrak{g} \to \mathfrak{g} \tag{A.5}$$

satisfying the Leibniz rule

$$D([x, y]) = [D(x), y] + [x, D(y)], \quad \forall x, y \in \mathfrak{g}. \tag{A.6}$$

If $\mathfrak{g}$ is a graded Lie algebra, then the derivation D is called *homogeneous of degree* k, if $D(\mathfrak{g}_n) \subset \mathfrak{g}_{n+k}$, for all $n \geq 0$.

The set of all derivations of $\mathfrak{g}$ is denoted by $\mathrm{Der}(\mathfrak{g})$. One can show that $\mathrm{Der}(\mathfrak{g})$ is again a Lie algebra over k with bracket given by the commutator of derivations

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1. \tag{A.7}$$

A.1.2 Universal enveloping algebras

Given an associative unital k-algebra A, one can endow A with the structure of a Lie algebra $\mathrm{Lie}(A)$, which has the same underlying k-module as A, and whose Lie bracket is given by

$$[a_1, a_2] := a_1 a_2 - a_2 a_1, \quad \forall a_1, a_2 \in A. \tag{A.8}$$

The construction of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is in some sense the inverse to this.

Definition A.1.2. The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is an associative, unital k-algebra together with a k-linear morphism $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$, which satisfies the following universal property: given any associative, unital k-algebra A together with a morphism

$$f : \mathfrak{g} \to \mathrm{Lie}(A) \tag{A.9}$$

of Lie algebras, there exists a unique morphism

$$\overline{f} : \mathcal{U}(\mathfrak{g}) \to A \tag{A.10}$$

of k-algebras such that $\overline{f} \circ \iota = f$.

Granting its existence, it is standard to show that $\mathcal{U}(\mathfrak{g})$ is unique (up to unique isomorphism). Explicitly, $\mathcal{U}(\mathfrak{g})$ may be constructed as the quotient of the tensor algebra

$$T(\mathfrak{g}) = \bigoplus_{k \geq 0} \mathfrak{g}^{\otimes k} \tag{A.11}$$

by the two-sided ideal, generated by all elements

$$\iota([x, y]) - (\iota(x) \otimes \iota(y) - \iota(y) \otimes \iota(x)), \quad x, y \in \mathfrak{g}, \tag{A.12}$$

where $\iota : \mathfrak{g} \to T(\mathfrak{g})$ denotes the canonical inclusion.

Example A.1.3. If $\mathfrak{g}$ is an abelian Lie algebra with vector space basis $(e_i)_{i\in I}$, then

$$U(\mathfrak{g}) \cong k[(e_i)_{i\in I}], \tag{A.13}$$

the k-algebra of polynomials in the variables e_i.

A.1.3 Filtrations on Lie algebras

Every Lie algebra $\mathfrak{g}$ is endowed with two descending filtrations. On the one hand, we have the *lower central series* $\{\mathfrak{g}_n\}_{n\geq 0}$, defined inductively by

$$\mathfrak{g}_0 := \mathfrak{g}, \quad \mathfrak{g}_{n+1} := [\mathfrak{g}_n, \mathfrak{g}],\ n \geq 0, \tag{A.14}$$

and the *derived series*, defined inductively by

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(n+1)} := [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}],\ n \geq 0. \tag{A.15}$$

From the definition, it is clear that both $\mathfrak{g}^{(1)}$ and $\mathfrak{g}_1$ are equal to the commutator $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$, however, in general the two filtrations are rather different.

A.1.4 Free Lie algebras

Let X be a set, and denote by A_X the free, non-associative, unital k-algebra on the set X ([72], LA 4.2). The algebra A_X is graded for the degree, where all elements of X have degree one.

Definition A.1.4. The *free Lie algebra* $\mathbb{L}(X)$ on X is defined to be the quotient of A_X by the two-sided ideal I generated by all elements aa and $a(bc) + c(ab) + b(ca)$, where $a, b, c \in A_X$.

Since I is generated by homogeneous elements, it follows that $\mathbb{L}(X)$ inherits a grading from A_X.
Now denote by Ass_X the free, associative, unital k-algebra on X ([72], LA 4.4). It consists of all k-linear combinations of elements of the free monoid X^* (i.e. the set of all words on the alphabet X, endowed with the concatenation product), and the multiplication on X^* is extended k-linearly to Ass_X. In fact, Ass_X is even a Hopf algebra [1] whose coproduct Δ is the unique coproduct, such that all $x \in X$ are primitive, i.e. $\Delta(x) = 1 \otimes x + x \otimes 1$, and whose antipode S is the unique anti-homomorphism of Ass_X, satisfying $S(x) = -x$. Moreover, Ass_X is graded for the degree, where every $x \in X$ has degree one.

Proposition A.1.5. *The universal enveloping algebra $\mathcal{U}(\mathbb{L}(X))$ is isomorphic to Ass_X, the free, associative, unital algebra on X.*

Proof: [72], Theorem 4.2. □

A.1.5 Shuffle algebras

For a set X, the *shuffle algebra* $k\langle X\rangle$ [66] has the same underlying k-module than Ass_X, but the multiplication is given by the shuffle product ⧢, defined recursively as follows

$$w \shuffle 1 = 1 \shuffle w = w, \quad \forall w \in X^*$$
$$x_i v' \shuffle x_j w' = x_i(v' \shuffle x_j w') + x_j(x_i v' \shuffle w'), \quad \forall x_i, x_j \in X, \forall v', w' \in X^*. \tag{A.16}$$

One can show that the shuffle product is both commutative and associative, with neutral element the empty word 1. In other words, $k\langle X\rangle$ is a commutative, associative unital k-algebra. It also carries a natural coproduct

$$\Delta : k\langle X\rangle \to k\langle X\rangle \otimes k\langle X\rangle \tag{A.17}$$

which is given by "deconcatenation"

$$\Delta(x_{i_1} \dots x_{i_n}) = \sum_{j=0}^{n} x_{i_1} \dots x_{i_j} \otimes x_{i_{j+1}} \dots x_{i_n}. \tag{A.18}$$

In the special case where X is finite (the most interesting case for us in this thesis), one can show (cf. [66], Chapter I) that $k\langle X\rangle$ is the dual bialgebra of Ass_X. More precisely, denoting by

$$(Ass_X)^\vee := \bigoplus_{n\geq 0} (Ass_X^n)^\vee, \tag{A.19}$$

the graded dual of Ass_X, we have a natural isomorphism

$$k\langle X\rangle \cong (Ass_X)^\vee. \tag{A.20}$$

A.1.6 Completion of free Lie algebras

Recall from Appendix A.1.3 the definition of the lower central series $\{\mathfrak{g}_n\}$ on a Lie algebra $\mathfrak{g}$. For every $m \geq n$, we have canonical morphisms

$$\mathfrak{g}/\mathfrak{g}_m \to \mathfrak{g}/\mathfrak{g}_n, \tag{A.21}$$

which form an inverse system of Lie algebras.

Definition A.1.6. Define the completion $\widehat{\mathbb{L}}(X)$ of $\mathbb{L}(X)$ to be the inverse limit

$$\varprojlim_n \mathbb{L}(X)/\mathbb{L}(X)_n, \tag{A.22}$$

of the above inverse system.

The Lie algebra $\widehat{\mathbb{L}(X)}$ endowed with the inverse limit topology is a topological Lie algebra. By definition, every element $f \in \widehat{\mathbb{L}(X)}$ can be represented uniquely by an infinite sum

$$f = \sum_{n=0}^{\infty} f_n, \quad f_n \in (\mathbb{L}(X)_n \setminus \mathbb{L}(X)_{n+1}) \cup \{0\}, \tag{A.23}$$

and the Lie bracket on $\widehat{\mathbb{L}(X)}$ is given by

$$\widehat{[f,g]} = h = \sum_{n\geq 0} h_n, \quad h_n = \sum_{k=0}^{n} [f_k, g_{n-k}] \in \mathbb{L}(X)_n. \tag{A.24}$$

The free associative unital algebra Ass_X can also be completed in a similar way, using the *augmentation ideal* $I(X)$ of Ass_X. By definition, $I(X) \subset Ass_X$ is the two-sided ideal generated by all elements $x \in X$. The powers of $I(X)$ define an inverse system of k-algebras via the canonical maps

$$Ass_X/I(X)^m \to Ass_X/I(X)^n, \quad m \geq n, \tag{A.25}$$

and one defines

$$\widehat{Ass}_X := \varprojlim_n Ass_X/I(X)^n. \tag{A.26}$$

The Hopf algebra structure of Ass_X passes to $\widehat{Ass}_X$: more precisely, $\widehat{Ass}_X$ is a topological Hopf algebra, which will sometimes also be denoted by $k\langle\langle X\rangle\rangle$. It is the linear dual space of the shuffle algebra $k\langle X\rangle$ [66].

For the rest of this section, we assume that k is a $\mathbb{Q}$-algebra. There exists a duality between *group-like elements* of $k\langle\langle X\rangle\rangle$ and $\mathbb{Q}$-algebra homomorphisms $\mathbb{Q}\langle X\rangle \to k$. By definition, $f \in k\langle\langle X\rangle\rangle$ is group-like, if (f_1 denoting the constant term of f)

$$f_1 = 1, \quad \Delta f = f \otimes f, \tag{A.27}$$

where Δ denotes the (completed) coproduct on $k\langle\langle X\rangle\rangle$, which is induced from the coproduct on Ass_X.

Proposition A.1.7. *Let k be a $\mathbb{Q}$-algebra and $f \in k\langle\langle X\rangle\rangle$ be a group-like series. Then the induced morphism*

$$\begin{aligned} \mathbb{Q}\langle X\rangle &\to k \\ w &\mapsto f_w, \end{aligned} \tag{A.28}$$

where f_w denotes the coefficient of w in the series f, is a homomorphism of $\mathbb{Q}$-algebras. Conversely, if $g : \mathbb{Q}\langle X\rangle \to k$ is a homomorphism of $\mathbb{Q}$-algebras, then the series

$$\sum_{w\in X^*} g(w) \cdot w \tag{A.29}$$

is group-like.

Proof: [66], Section 1.5. □

Corollary A.1.8. *Let $f \in k\langle\langle X\rangle\rangle$ be a group-like series. Then the $\mathbb{Q}$-linear span of the coefficients of f*

$$\mathrm{Span}_{\mathbb{Q}}\{f_w \mid w \in X^*\} \tag{A.30}$$

is a $\mathbb{Q}$-subalgebra of k.

Now let $\widehat{I(X)}k\langle\langle X\rangle\rangle$ be the two-sided ideal, which is topologically generated by X. Then we have exponential and logarithm maps

$$\exp : \widehat{I(X)} \to 1 + \widehat{I(X)}, \quad \log : 1 + \widehat{I(X)} \to \widehat{I(X)}, \tag{A.31}$$

which are inverse to one-another and are defined by the usual power series

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}, \quad \log(1+x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n!} \tag{A.32}$$

(recall that k was assumed to be a $\mathbb{Q}$-algebra).

Proposition A.1.9. *The map* exp *defines a bijection between $\widehat{\mathbb{L}(X)}$ and the set of group-like elements of $k\langle\langle X\rangle\rangle$*

$$\{f \in 1 + \widehat{I(X)} \mid \Delta(f) = f \otimes f\}. \tag{A.33}$$

Proof: [72], Corollary 7.3. □

A.2 Iterated integrals and linear differential equations

In this section, we mostly follow [46]. A good reference is also [19].

A.2.1 Definition and properties

Let k denote either $\mathbb{R}$ or $\mathbb{C}$, and let M be a smooth manifold over k. The set of all piecewise smooth paths on M, we denote by PM, while $E^1(M)$ denotes the k-vector space of all smooth differential one-forms on M.

Definition A.2.1. Let $\gamma \in PM$ and $\omega_1, \ldots, \omega_n \in E^1(M)$. Define the *iterated* integral $\int_\gamma \omega_1 \ldots \omega_n$ by the formula

$$\int_\gamma \omega_1 \ldots \omega_n = \int\cdots\int_{0\leq t_1\leq \ldots t_n\leq 1} f_1(t_1)\ldots f_n(t_n)\mathrm{d}t_1 \ldots \mathrm{d}t_n, \tag{A.34}$$

where $f_i(t_i)\mathrm{d}t_i = \gamma^*(\omega_i)$.

The following proposition subsumes many of the useful properties, which are satisfied by iterated integrals.

Proposition A.2.2. *Iterated integrals satisfy the following properties.*

(i) *If γ_1, γ_2 are two composable paths, i.e. $\gamma_1(1) = \gamma_2(0)$, then*

$$\int_{\gamma_1 \circ \gamma_2} \omega_1 \dots \omega_n = \sum_{i=0}^{n} \int_{\gamma_1} \omega_1 \dots \omega_i \int_{\gamma_2} \omega_{i+1} \dots \omega_n. \tag{A.35}$$

(ii) *Denote by $\gamma^{-1} : [0,1] \to M$ the reversed path $\gamma^{-1}(t) := \gamma(1-t)$. Then*

$$\int_{\gamma^{-1}} \omega_1 \dots \omega_n = (-1)^n \int_{\gamma} \omega_n \dots \omega_1. \tag{A.36}$$

(iii) *The product of iterated integrals is given by the shuffle product, i.e.*

$$\int_{\gamma} \omega_1 \dots \omega_r \int_{\gamma} \omega_{r+1} \dots \omega_{r+s} = \sum_{\sigma \in \Sigma(r,s)} \int_{\gamma} \omega_{\sigma(1)} \dots \omega_{\sigma(r+s)}, \tag{A.37}$$

where

$$\Sigma(r,s) = \{\sigma \in \Sigma(r+s) \,|\, \sigma^{-1}(1) < \dots < \sigma^{-1}(r),\, \sigma^{-1}(r+1) < \dots < \sigma^{-1}(r+s)\}. \tag{A.38}$$

(iv) *The iterated integral $\int_\gamma \omega_1 \dots \omega_n$ is invariant under reparametrizations of γ.*

(v) *Iterated integrals are functorial with respect to smooth maps, i.e. if $f : N \to M$ is a smooth map, $\gamma \in PN$ and $\omega_1, \dots, \omega_n \in E^1(M)$, then*

$$\int_{\gamma} f^*(\omega_1) \dots f^*(\omega_n) = \int_{f_*(\gamma)} \omega_1 \dots \omega_n, \tag{A.39}$$

where $f_*(\gamma) := f \circ \gamma$.

(vi) *For a path $\gamma : [0,1] \to M$, and $a, b \in [0,1]$ with $a < b$, let $\gamma_a^b : [0,1] \to M$ be the path $\gamma_a^b(s) = \gamma(bs + (1-s)a)$, which starts at a and ends at b. Then for all smooth differential one-forms $\omega_1, \dots, \omega_n$, we have*

$$\frac{\partial}{\partial t}\Big|_{t=a} \int_{\gamma_t^b} \omega_1 \dots \omega_n = -\langle \omega_1, \gamma'(a) \rangle \left(\int_{\gamma_a^b} \omega_2 \dots \omega_n \right), \tag{A.40}$$

$$\frac{\partial}{\partial t}\Big|_{t=b} \int_{\gamma_a^t} \omega_1 \dots \omega_n = \left(\int_{\gamma_a^b} \omega_1 \dots \omega_{n-1} \right) \langle \omega_n, \gamma'(b) \rangle, \tag{A.41}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between differential forms and tangent vectors.

Proof: The statements (i)-(v) are given in [46], as Proposition 2.9, Proposition 2.12, Lemma 2.11, Proposition 2.4(a) and Proposition 1.2 respectively. On the other hand, (vi) is given in [42] as Exercise 36. For the sake of completeness, we give a proof.
For $t \in [0,1]$ with $t \leq b$, we can write $\gamma_t^b = \gamma_{|[t,b]} \circ \varphi_t^b$, where $\varphi_t^b(s) = sb + (1-s)t$. Then by the very definition of iterated integrals, we get

$$\int_{\gamma_t^b} \omega_1 \dots \omega_n = \int\cdots\int_{0\leq t_1 \leq \dots \leq t_n \leq 1} (\gamma_t^b)^*(\omega_1)\dots(\gamma_t^b)^*(\omega_n). \tag{A.42}$$

Using the functoriality of iterated integrals and invariance under reparametrization of paths, we obtain that (A.42) equals

$$\int\cdots\int_{t\leq t_1 \leq \dots \leq t_n \leq b} \gamma^*(\omega_1)\dots\gamma^*(\omega_n). \tag{A.43}$$

Using the fundamental theorem of calculus, we now get (A.40). The proof of (A.41) is analogous. □

Note that iterated integrals define functions on the set PM of all piecewise smooth paths of M by the assignment

$$\begin{aligned} \int \omega_1 \dots \omega_n : PM &\to k \\ \gamma &\mapsto \int_\gamma \omega_1 \dots \omega_n. \end{aligned} \tag{A.44}$$

A.2.2 Linear differential equations and exponentials

The following classical proposition shows that solutions to initial value problems (IVPs) are given by exponentials of Lie series (Proposition A.1.9). We cite from [43], Proposition 4.1 (with very minor notational differences). We use the notation and terminology of Appendix A.1 in the case $k = \mathbb{C}$.

Proposition A.2.3. *Suppose that* $A : [0,1] \to \widehat{\mathbb{L}(X)}$ *is smooth, i.e. every coefficient of* A *is a smooth function of* $t \in [0,1]$*. If* $f : [0,1] \to \mathbb{C}\langle\langle X \rangle\rangle$ *satisfies the IVP*

$$f' = A \cdot f, \quad f(a) = 1, \tag{A.45}$$

then $f(t)$ *is group-like for all* $t \in [0,1]$*.*

Proof: Using that Δ is a $\mathbb{Q}$-algebra homomorphism, and $A(t)$ is a Lie-series for every t, we see that

$$(\Delta f)' = \Delta(f') = \Delta(Af) = (\Delta A)(\Delta f) = (A \otimes 1 + 1 \otimes A)(f \otimes f). \tag{A.46}$$

On the other hand, by the product rule

$$(f \otimes f)' = f' \otimes f + f \otimes f' = A \cdot f \otimes f + f \otimes A \cdot f = (A \otimes 1 + 1 \otimes A) \cdot (f \otimes f). \quad \text{(A.47)}$$

Thus both $\Delta(f)$ and $f \otimes f$ satisfy the IVP

$$g' = (A \otimes 1 + 1 \otimes A) \cdot g, \quad Y(0) = 1 \otimes 1, \quad \text{(A.48)}$$

where g is a function $g : [0,1] \to \mathbb{C}\langle\langle X \rangle\rangle \otimes \mathbb{C}\langle\langle X \rangle\rangle$. But since solutions to IVPs are unique, it follows that $\Delta(f)(t) = f(t) \otimes f(t)$ for all t. □

A.2.3 Homotopy invariance

A particularly important subclass of all iterated integrals are the ones, which are invariant under path homotopies. In general, a function $\Phi : PM \to k$ is called a *homotopy functional*, if $\Phi(\gamma_0) = \Phi(\gamma_1)$ for all pairs of paths γ_0, γ_1, which are homotopic relative to their endpoints.

Definition A.2.4. An iterated integral $\int \omega_1 \ldots \omega_n$ is *homotopy invariant*, if the induced map on PM (A.44) is a homotopy functional. We define $H^0(\mathbb{B}(M))$ to be the set of all homotopy invariant iterated integrals on M.

The set $H^0(\mathbb{B}(M))$ was introduced and studied by Chen in [25] as the zeroth cohomology group of the so-called (reduced) bar complex $\mathbb{B}(M)$ of M, whence the notation. It is a commutative k-algebra with the multiplication given by the shuffle product of iterated integrals, and even a commutative Hopf algebra over k.

Example A.2.5. Let $M = \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{C})$. Then by Chen's π_1-de Rham theorem [25, 42, 46], a vector space basis for $H^0(\mathbb{B}(M))$ is given by the family of all iterated integrals $\int \omega_{i_1} \ldots \omega_{i_n}$, with $\omega_{i_j} = \frac{\mathrm{d}z}{z - i_j}$ for $i_j \in \{0, 1\}$. The non-commutative generating series

$$T(\gamma) := \sum_{w = x_{i_1} \ldots x_{i_n} \in \langle x_0, x_1 \rangle} \int_\gamma \omega_{i_1} \ldots \omega_{i_n} w \quad \text{(A.49)}$$

then defines for any choice of base points $a, b \in M$ a $\mathbb{C}$-linear morphism

$$\begin{aligned} T : \mathbb{C}[\pi_1(M; a, b)] &\to \mathbb{C}\langle\langle x_0, x_1 \rangle\rangle \\ \gamma &\mapsto T(\gamma), \end{aligned} \quad \text{(A.50)}$$

which is injective and has dense image (cf. [42], Section 1.7).

A.2.4 Regularization and tangential base points

We now specialize to the case where $k = \mathbb{C}$ and the manifold M is one-dimensional, in other words, a Riemann surface. We will write C (for "curve") instead of M from now on. Let $S \subset C$ be a discrete subset. Denote by $\Omega^1(C; \log(S))$ the $\mathbb{C}$-vector space of meromorphic differential one-forms on $\mathbb{C}$, which are holomorphic on $C \setminus S$, and which have at most logarithmic poles at every point $s \in S$. This last condition means that for every $\omega \in \Omega^1(C; \log(S))$ and every $s \in S$, there exists a local coordinate z_s at s, such that ω written in that local coordinate equals

$$\frac{\mathrm{d}z_s}{z_s}. \tag{A.51}$$

For two points $s \in S$ and $b \in C \setminus S$, let $P(C; s, b)$ be the set of all paths γ from s to b, such that $\gamma((0, 1]) \subset C \setminus S$. Although the naive iterated integral

$$\int_\gamma \omega_1 \dots \omega_n, \quad \omega_i \in \Omega^1(C; \log(S)) \tag{A.52}$$

will diverge in general, in [26], §15, Deligne describes a procedure for regularizing the iterated integral (A.52) so that it converges. We will describe in more detail a special case of this regularization, which is sufficient for our purposes.

Let X be an alphabet, and $\mathbb{C}\langle\langle X\rangle\rangle$ as defined in the last section. Let $\omega \in \Omega^1(C)\widehat{\otimes} I(X)$ be an $I(X)$-valued, formal differential one-form. Then for every path $\gamma \in PM$, the formal series

$$\exp\left[\int_\gamma \omega\right] := 1 + \sum_{k\geq 1} \int_\gamma \omega^k \tag{A.53}$$

is a well-defined element of $\mathbb{C}\langle\langle X\rangle\rangle$. By Proposition A.2.2.(vi), the function

$$T_\omega(\gamma_t^1) := \exp\left[\int_{\gamma_t^1} \omega\right], \quad t \in [0, 1] \tag{A.54}$$

satisfies the differential equation

$$\frac{\partial}{\partial t} f(t) = -\omega \cdot f(t). \tag{A.55}$$

In particular, if $\gamma \in P(C; s, b)$ is a path as above, and $\omega \in \Omega^1(C; \log(S))\widehat{\otimes} I(X)$, then near $t = 0$ (A.55) becomes

$$\frac{\partial}{\partial t} f(t) = -\frac{\nabla(\omega)_s}{t} \cdot f(t) \tag{A.56}$$

where $\nabla(\omega)_s \in I(X)$ is the residue of ω at s, which induces, by left multiplication, a $\mathbb{C}$-linear endomorphism of $\mathbb{C}\langle\langle X\rangle\rangle$.

Since ω takes values in $I(X)$, the image $[\nabla(\omega)_s]_n$ of $\nabla(\omega)_s$ in the quotient $\mathbb{C}\langle\langle X\rangle\rangle/I(X)^n$ is a nilpotent endomorphism of $\mathbb{C}\langle\langle X\rangle\rangle/I(X)^n$, for every $n \geq 1$. Hence, from the general theory of linear differential equations with nilpotent residues (cf. [77], Ch. II), it follows that the limit

$$\lim_{t\to 0}[t^{\nabla(\omega)_s} T_\omega(\gamma_t^1)]_n \tag{A.57}$$

exists for any $n \geq 0$. Also, for varying n, the elements (A.57) are compatible with the morphisms $\mathbb{C}\langle\langle X\rangle\rangle/I(X)^m \to \mathbb{C}\langle\langle X\rangle\rangle/I(X)^n$, for $m \geq n$. We summarize this discussion in the following

Proposition A.2.6. *For ω, γ and $a, b \in C \setminus S$ as above, the limit*

$$\lim_{t\to 0} t^{\nabla(\omega)_s} T_\omega(\gamma_t^b) \in \mathbb{C}\langle\langle X\rangle\rangle \tag{A.58}$$

exists. Similarly, the limit

$$\lim_{t\to 0} T_\omega(\gamma_a^t) t^{-\nabla(\omega)_s} \in \mathbb{C}\langle\langle X\rangle\rangle \tag{A.59}$$

exists.

Implicitly, the limits (A.58), (A.59) depend also on the choice of coordinate t. In [26], this ambiguity is resolved by fixing a non-zero tangent vector $\vec{v}_s \in T_s(C)^\times$ at s, and demanding in addition that $\gamma'(0) = \vec{v}_s$. On the other hand, the mere existence of the limits is unaffected by the choice of tangent vector.

Example A.2.7. On $C = \mathbb{P}^1 \setminus \{\infty\}$ with canonical coordinate z, consider the differential one-form

$$\omega_{\mathrm{KZ}} = \frac{\mathrm{d}z}{z} x_0 + \frac{\mathrm{d}z}{z-1} x_1 \in \Omega^1(C; \log(S)), \quad S = \{0, 1\}. \tag{A.60}$$

This differential form has residue x_0 at 0 and x_1 at 1.

Fix the tangent vectors $\vec{1}_0 = \frac{\partial}{\partial z} \in T_0(C)^\times$ and $\vec{1}_1 = \frac{\partial}{\partial z} \in T_1(C)$. Then the straight line path $\gamma : [0,1] \hookrightarrow \mathbb{C}$ from 0 to 1 satisfies $\gamma'(0) = \vec{1}_0$ and $\gamma'(1) = \vec{1}_1$. By Proposition A.2.6, it follows that the limit

$$\lim_{t\to 0} t^{x_0} \exp\left[\int_{\gamma_t^{1-t}} \omega_{\mathrm{KZ}}\right] t^{-x_1} \tag{A.61}$$

exists. This is essentially the Drinfeld associator, more precisely (A.61) equals

$$\Phi(x_0, x_1)^{op} \in \mathbb{C}\langle\langle X\rangle\rangle^{op}, \tag{A.62}$$

where a supscript *op* denotes the opposite algebra, i.e. concatenation of words is reversed: $(x_0x_1)^{op} = x_1x_0$.

Remark A.2.8. As noted above, the theory presented here is but a very special case of Deligne's theory of the "π_1 à points bases tangentielles" (cf. [26], §15). At the heart of this theory is the definition of the fundamental group $\pi_1(X; \vec{v})$, where X is a smooth algebraic curve and $\vec{v}$ is a tangential base point in a suitable "motivic" sense.

Appendix B

Linear independence of indefinite iterated Eisenstein integrals

LINEAR INDEPENDENCE OF INDEFINITE ITERATED EISENSTEIN INTEGRALS

NILS MATTHES

Abstract. We prove linear independence of indefinite iterated Eisenstein integrals over the fraction field of the ring of formal power series $\mathbb{Z}[[q]]$. Our proof relies on a general criterium for linear independence of iterated integrals, which has been established by Deneufchâtel, Duchamp, Minh and Solomon. As a corollary, we obtain $\mathbb{C}$-linear independence of indefinite iterated Eisenstein integrals, which has applications to the study of elliptic multiple zeta values, as defined by Enriquez.

1. Introduction

Given a collection $\omega_1, \ldots, \omega_r$ of smooth one-forms on a smooth manifold M, and a smooth path $\gamma : [0,1] \to M$, one defines their *iterated integral* as

$$\int_\gamma \omega_1 \ldots \omega_r = \int_{0 \leq t_1 \leq \ldots \leq t_r \leq 1} \gamma^*(\omega_1) \ldots \gamma^*(\omega_r), \tag{1.1}$$

where $\gamma^*(\omega_i) = f_i(t_i)\mathrm{d}t_i$ denotes the pull back of ω_i along γ. In the case of a single differential one-form ω, (1.1) is simply the path integral of ω along γ.

A classical application of iterated integrals is the construction of solutions to certain systems of linear differential equations via the method of Picard iteration (cf. e.g. [14]). However, iterated integrals also appear in number theory, prominent examples being *multiple polylogarithms* and *multiple zeta values*, which are iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (see for example the lecture notes [8] for an introduction from the point of view of iterated integrals). It is known that the multiple polylogarithms are linearly independent over $\mathbb{C}$ [13]. Using rather different techniques, this result has been generalized [5], with $\mathbb{C}$ replaced by an arbitrary field of functions satisfying some extra conditions.

On the other hand, another family of iterated integrals arising in number theory are iterated integrals of modular forms. Their study has been initiated by Manin [11], and was later extended in [3, 7, 9]. Known in the literature under the names *iterated Eichler integrals* [3] or *iterated Shimura integrals* [11], these are iterated integrals on the upper half-plane, which generalize the classical Eichler integrals [10], and are also closely related to L-functions of modular forms [11, 3].

Iterated integrals of modular forms also appear in the study of *elliptic multiple zeta values* [4, 6, 2, 12], the latter being a natural genus one analogue of the classical multiple zeta values. In [2], a procedure for decomposing elliptic multiple zeta values into certain $\mathbb{C}$-linear combinations of (indefinite) iterated integrals of Eisenstein series (called iterated Eisenstein integrals for short)[1] is described. The uniqueness of this decomposition, important both for the mathematical theory as well as for applications to physics [1], depended on the $\mathbb{C}$-linear independence of the iterated Eisenstein integrals in question.

In this paper, we prove linear independence of iterated Eisenstein integrals, first over the fraction field $\mathrm{Frac}(\mathbb{Z}[[q]])$ of the ring of formal power series in one variable with integer coefficients, where q is viewed as a coordinate on the open unit disk. By the main result of [5], it is enough to prove that $\mathrm{Frac}(\mathbb{Z}[[q]])$ does not contain primitives of Eisenstein series, which in turn follows from a computation of their denominators.

Having established linear independence over $\mathrm{Frac}(\mathbb{Z}[[q]])$, the linear independence of iterated Eisenstein integrals over $\mathbb{Q}$ follows immediately, since $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[[q]])$. Finally, by extending scalars from $\mathbb{Q}$ to $\mathbb{C}$, we obtain the desired $\mathbb{C}$-linear independence of iterated Eisenstein integrals.

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2. Iterated Eisenstein integrals

2.1. **Eisenstein series.** . For $k \geq 1$ denote by G_{2k} the *Hecke-normalized Eisenstein series* (cf. e.g. [17]), which is the function on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, defined by the convergent series

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n \in \mathbb{Q} \oplus q\mathbb{Z}[[q]], \quad q = e^{2\pi i \tau},$$

where B_{2k} denotes the $2k$-th Bernoulli number, and $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$. We also set $G_0 \equiv -1$.

The function G_{2k} is holomorphic, and, for $k \geq 2$, it is a modular form for $\mathrm{SL}_2(\mathbb{Z})$. Write G_{2k}^{∞} for the constant term in its q-expansion, and likewise $G_{2k}^0(q)$ for $G_{2k}(q) - G_{2k}^{\infty}$. Note that for $k \geq 1$, we have

$$G_{2k}^{\infty} = -\frac{B_{2k}}{4k}, \quad G_{2k}^0(q) = \sum_{n \geq 1} \sigma_{2k-1}(n) q^n.$$

2.2. **Regularization of iterated integrals.** . We would now like to define iterated Eisenstein integrals

$$\int_{\tau}^{i\infty} G_{2k_1}(q_1)\mathrm{d}\tau_1 \ldots G_{2k_n}(q_n)\mathrm{d}\tau_n$$

as functions depending on some start point $\tau \in \mathbb{H}$, where the integration is performed along some path from τ to the cusp $i\infty$[2]. Unfortunately, in this case the usual definition of iterated integrals (1.1) produces divergent integrals, already in the case of single Eisenstein integrals, i.e. for $n = 1$. In order to overcome this problem, we describe a regularization scheme for

[1] All modular forms appearing in this paper are modular forms for the group $\mathrm{SL}_2(\mathbb{Z})$.

[2] The value of the iterated integral does not depend on the choice of path, since the Eisenstein series are holomorphic functions on a one-dimensional complex manifold.

such iterated integrals, introduced by Brown in [3]. For the rest of this subsection, we follow [3].

Let $W = \mathbb{C}[[q]]^{<1}$ be the $\mathbb{C}$-algebra of formal power series, which converge on the open q-disk $D = \{q \in \mathbb{C} \mid |q| < 1\}$, and denote by $D^* := D \setminus \{0\}$ the punctured disk. Via the universal covering map

$$\exp : \mathbb{H} \to D^*, \quad \tau \mapsto e^{2\pi i \tau}, \tag{2.1}$$

we can consider W as a $\mathbb{C}$-subalgebra of the $\mathbb{C}$-algebra $\mathcal{O}(\mathbb{H})$ of holomorphic functions on the upper half-plane.

Write $W = W^0 \oplus W^\infty$ with $W^0 = q\mathbb{C}[[q]]$ and $W^\infty = \mathbb{C}$. For a power series $f \in W$, define f^0 to be its image in W^0 under the natural projection, and define $f^\infty \in W^\infty$ likewise. Denote by $T^c(W)$ the tensor coalgebra on the $\mathbb{C}$-vector space W, which comes equipped with a shuffle product $\sqcup\!\sqcup$. We will use bar notation for elements of $T^c(W)$, and define a map $R : T^c(W) \to T^c(W)$ by the formula

$$R[f_1|\dots|f_n] = \sum_{i=0}^{n} (-1)^{n-i}[f_1|\dots|f_i] \sqcup\!\sqcup [f_n^\infty|\dots|f_{i+1}^\infty].$$

We can now make the

Definition 2.1. Given $f_1, \dots, f_n \in W \subset \mathcal{O}(\mathbb{H})$ as above, their *regularized iterated integral* is defined as

$$I(f_1, \dots, f_n; \tau) := \sum_{i=0}^{n} \int_\tau^{i\infty} R[f_1|\dots|f_i]_{\mathrm{d}\tau} \int_\tau^0 [f_{i+1}^\infty|\dots|f_n^\infty]_{\mathrm{d}\tau}, \tag{2.2}$$

where

$$\int_a^b [f_1|\dots|f_n]_{\mathrm{d}\tau} := \int_a^b f_1(\tau_1)\mathrm{d}\tau_1 \dots f_n(\tau_n)\mathrm{d}\tau_n.$$

Proposition 2.2. *For all $f_1, \dots, f_n \in W$, $I(f_1, \dots, f_n; \tau)$ is well-defined, i.e.* (2.2) *is finite, and we have*

$$\frac{\partial}{\partial \tau}\Big|_{\tau=\tau_0} I(f_1, \dots, f_n; \tau) = -f_1(\tau_0) I(f_2, \dots, f_n; \tau_0).$$

Proof: [3], Lemma 4.5 and Proposition 4.7 i). □

The second part of the preceding proposition is the analogue for regularized iterated integrals of the differential equation satisfied by ordinary iterated integrals ([8], p.40). It will be crucial in the proof of linear independence of iterated Eisenstein integrals.

2.3. **Iterated integrals on the q-disk.** . We have seen that $I(f_1, \dots, f_n; \tau)$ is a holomorphic function on the upper half-plane. Using the change of coordinates (2.1), we can rewrite $I(f_1, \dots, f_n; \tau)$ as a regularized iterated integral on the punctured q-disk

$$I(f_1, \dots, f_n; \tau) = \frac{1}{(2\pi i)^n} \sum_{i=0}^{n} \int_q^0 R[f_1|\dots|f_i]_{\frac{\mathrm{d}q}{q}} \int_q^1 [(f^\infty)_{i+1}|\dots|(f^\infty)_n]_{\frac{\mathrm{d}q}{q}}. \tag{2.3}$$

The virtue of representation (2.3) is that one sees that

$$I(f_1,\dots,f_n;\tau)\in W[\log(q)],\quad \log(q):=2\pi i\tau,$$

and therefore every linear identity between the $I(f_1,\dots,f_n;\tau)$ reduces, by comparing coefficients, to a linear system of equations. Also, note that if all $f_i\in W_{\mathbb{Q}}:=\mathbb{Q}[[q]]\cap W$, then $(2\pi i)^nI(f_1,\dots,f_n;\tau)\in W_{\mathbb{Q}}[\log(q)]$.

Definition 2.3. For $k_1,\dots,k_n\geq 0$, we define the *(indefinite, Hecke-normalized) iterated Eisenstein integral* to be

$$\mathcal{G}(2k_1,\dots,2k_n;q)=(2\pi i)^nI(G_{2k_1},\dots,G_{2k_n};\tau)\in W_{\mathbb{Q}}[\log(q)]. \tag{2.4}$$

Note that by Proposition 2.2,

$$\begin{aligned}\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\Big|_{\tau=\tau_0}\mathcal{G}(2k_1,\dots,2k_n;q)&=q\frac{\partial}{\partial q}\Big|_{q=q_0}\mathcal{G}(2k_1,\dots,2k_n;q)\\&=-G_{2k_1}(q_0)\mathcal{G}(2k_2,\dots,2k_n;q_0).\end{aligned} \tag{2.5}$$

Example 2.4. In length one, we have (cf. [3], Example 4.10)

$$\mathcal{G}(2k;q)=\frac{B_{2k}}{4k}\log(q)-\sum_{n\geq1}\frac{\sigma_{2k-1}(n)}{n}q^n.$$

Later on, we will also need the integral over the non-constant part G^0_{2k} of the Eisenstein series G_{2k}. We denote this by

$$\mathcal{G}^0(2k;q):=\int_q^0G^0_{2k}(q_1)\frac{\mathrm{d}q_1}{q_1}=-\sum_{n\geq1}\frac{\sigma_{2k-1}(n)}{n}q^n. \tag{2.6}$$

3. Proof of linear independence

Having defined iterated Eisenstein integrals in the last section, we now turn to the proof of their linear independence. The larger part of this section is devoted to proving linear independence over $\mathrm{Frac}(\mathbb{Z}[[q]])$, the fraction field of the ring of formal power series with integer coefficients. In order to achieve this, we use the following general linear independence result for iterated integrals, which is (a special case of) the main result of [5] (Theorem 2.1). Let X be an alphabet (not necessarily finite), and denote by X^* the free monoid on X.

Theorem 3.1 (Deneufchâtel, Duchamp, Minh, Solomon)**.** *Let $(\mathcal{A},\mathrm{d})$ be a differential algebra over a field k of characteristic zero, whose ring of constants* $\ker(\mathrm{d})$ *is precisely equal to k. Let $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. a subfield such that $\mathrm{d}\mathcal{C}\subset\mathcal{C}$). Suppose that $S\in\mathcal{A}\langle\langle X\rangle\rangle$ is a solution to the differential equation*

$$\mathrm{d}S=M\cdot S,$$

where $M=\sum_{x\in X}u_xx\in\mathcal{C}\langle\langle X\rangle\rangle$ is a homogeneous series of degree 1, with initial condition $S_1=1$, where S_1 denotes the coefficient of the empty word in the series S. The following are equivalent:

1. *The family of coefficients $(S_w)_{w\in X^*}$ of S is linearly independent over $\mathcal{C}$.*

(2) *The family* $\{u_x\}_{x\in X}$ *is linearly independent over* k*, and we have*

$$\mathrm{d}\mathcal{C} \cap \mathrm{Span}_k(\{u_x\}_{x\in X}) = \{0\}. \tag{3.1}$$

We are now in a position to prove our main result.

Theorem 3.2. *The family of iterated Eisenstein integrals* (2.4) *is linearly independent over* Frac($\mathbb{Z}[[q]]$).

Proof: We will apply Theorem 3.1 with the following parameters:

- $k = \mathbb{Q}$, $\mathcal{A} = \mathbb{Q}[\log(q)]((q))$ with differential $\mathrm{d} = q\frac{\partial}{\partial q}$, and $\mathcal{C} = \mathrm{Frac}(\mathbb{Z}[[q]])$ (the latter is a differential field by the quotient rule for derivatives)
- $X = \{a_{2k}\}_{k\geq 0}$, $u_{a_{2k}} = -G_{2k}(q)$, hence

$$M(\tau) = -\sum_{k\geq 0} G_{2k}(q)a_{2k}.$$

With these conventions, it follows from (2.5) that the formal series

$$1 + \int_\tau^{i\infty} [M(\tau_1)]_{\mathrm{d}\tau} + \int_\tau^{i\infty} [M(\tau_1)|M(\tau_2)]_{\mathrm{d}\tau} + \ldots \in \mathcal{O}(\mathbb{H})\langle\langle X\rangle\rangle,$$

where the iterated integrals are regularized as in Section 2.2, is a solution to the differential equation $\mathrm{d}S = M \cdot S$, with $S_1 = 1$. Consequently, the coefficient of the word $w = a_{2k_1}\ldots a_{2k_n}$ in S is equal to $\mathcal{G}(2k_1,\ldots,2k_n;q)$. Moreover, since $\mathbb{Q}$-linear independence of the Eisenstein series is well-known (cf. e.g. [16], VII.3.2), it remains to verify (3.1) in our situation.

To this end, assume that there exist $\alpha_{2k} \in \mathbb{Q}$, all but finitely many of which are equal to zero, such that

$$\sum_{k\geq 0} \alpha_{2k}G_{2k}(q) \in \mathrm{d}\mathcal{C}. \tag{3.2}$$

Clearing denominators, we may assume that $\alpha_{2k} \in \mathbb{Z}$. Furthermore, from the definition of $\mathrm{d} = q\frac{\partial}{\partial q}$, one sees that the image $\mathrm{d}\mathcal{C}$ of the differential operator d does not contain any constant except for zero. Therefore, the coefficient of the word 1 in (3.2) vanishes; in other words

$$\sum_{k\geq 0} \alpha_{2k}G_{2k}(q) = \sum_{k\geq 1} \alpha_{2k}G^0_{2k}(q) \in q\mathbb{Q}[[q]].$$

Now the differential d is invertible on $q\mathbb{Q}[[q]]$, and inverting d is the same as integrating. Hence (3.2) is equivalent to

$$\sum_{k\geq 1} \alpha_{2k}\mathcal{G}^0(2k;q) \in \mathcal{C}. \tag{3.3}$$

But this is absurd, unless all the α_{2k} vanish, as we shall see now. Indeed, if $f \in \mathcal{C} = \mathrm{Frac}(\mathbb{Z}[[q]])$, then there exists $m \in \mathbb{Z}\setminus\{0\}$ such that $f \in \mathbb{Z}[m^{-1}]((q))$. This follows from the well-known inversion formula for power series. On the other hand, the coefficient of q^p in $\mathcal{G}^0(2k;q)$, for p a prime number, is given by

$$\frac{\sigma_{2k-1}(p)}{p} = \frac{p^{2k-1}+1}{p} \equiv \frac{1}{p} \mod \mathbb{Z}$$

(cf. (2.6)). Thus, we must have

$$\frac{1}{p}\sum_{k\geq 1}\alpha_{2k} \in \mathbb{Z}[m^{-1}],$$

for every prime number p. But then the integer $\sum_{k\geq 1}\alpha_{2k}$ is divisible by infinitely many primes (namely, at least all the primes which don't divide m), which implies $\sum_{k\geq 1}\alpha_{2k} = 0$.

Now assume that k_1 is the smallest positive, even integer with the property that $\alpha_{k_1} \neq 0$. Consider the coefficient of $q^{p^{k_1}}$ in $\mathcal{G}^0(2k;q)$, which is equal to

$$\frac{\sigma_{2k-1}(p^{k_1})}{p^{k_1}} = \frac{1}{p^{k_1}}\sum_{j=0}^{k_1} p^{j(2k-1)} \equiv \begin{cases} \frac{1}{p^{k_1}} \mod \mathbb{Z} & \text{if } 2k > k_1 \\ \frac{1}{p^{k_1}} + \frac{1}{p} \mod \mathbb{Z} & \text{if } 2k = k_1 \end{cases}$$

(cf. (2.6)). By (3.3), $\frac{\alpha_{k_1}}{p} + \frac{1}{p^{k_1}}\sum_{k\geq 1}\alpha_{2k} \in \mathbb{Z}[m^{-1}]$, and by what we have seen before, $\sum_{k\geq 1}\alpha_{2k} = 0$. Hence

$$\frac{\alpha_{k_1}}{p} \in \mathbb{Z}[m^{-1}],$$

for every prime number p, which again implies $\alpha_{k_1} = 0$, in contradiction to our assumption $\alpha_{k_1} \neq 0$. Therefore, in (3.3), we must have $\alpha_{2k} = 0$ for all $k \geq 1$ and (3.1) is verified. □

Having established linear independence of iterated Eisenstein integrals over the field $\mathrm{Frac}(\mathbb{Z}[[q]])$, linear independence over $\mathbb{C}$ follows almost immediately.

Corollary 3.3. *The family of iterated Eisenstein integrals $\mathcal{G}(2k_1,\ldots,2k_n;q)$ for $n \geq 0$ and all $k_i \geq 0$ is linearly independent over the complex numbers.*

Proof: Let $\mathcal{G}_1,\ldots,\mathcal{G}_n$ be iterated Eisenstein integrals, and assume we have a relation

$$\sum_{i=1}^{n}\alpha_i\mathcal{G}_i = 0$$

with $\alpha_i \in \mathbb{C}$. Since $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[[q]])$, it follows from Theorem 3.2 that the matrix of the coefficients of the $\mathcal{G}_i$, considered as series in $\log(q)^k q^l$ for $k,l \geq 0$, has maximal rank n. Therefore $\alpha_i = 0$ for $i = 1,\ldots,k$. □

Remark 3.4. By the shuffle product formula, the $\mathbb{C}$-vector space spanned by the iterated Eisenstein integrals is a $\mathbb{C}$-algebra. Corollary 3.3 implies that it is a free shuffle algebra, and thus a polynomial algebra by [15].

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Appendix C

Elliptic multiple zeta values and one-loop superstring amplitudes

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Elliptic multiple zeta values and one-loop superstring amplitudes

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Abstract

We investigate iterated integrals on an elliptic curve, which are a natural genus-one generalization of multiple polylogarithms. These iterated integrals coincide with the multiple elliptic polylogarithms introduced by Brown and Levin when constrained to the real line. At unit argument they reduce to an elliptic analogue of multiple zeta values, whose network of relations we start to explore. A simple and natural application of this framework are one-loop scattering amplitudes in open superstring theory. In particular, elliptic multiple zeta values are a suitable language to express their low energy limit. Similar to the techniques available at tree-level, our formalism allows to completely automatize the calculation.

1 Introduction

In recent years, we have witnessed numerous fruitful interactions between number theory and particle physics. A particularly rich domain of intersection are iterated integrals, which prominently appear in scattering amplitudes in field theories and string theories. For a large class of Feynman and worldsheet integrals, multiple polylogarithms were recognized as a suitable language to cast results into a manageable form, see e.g. refs. [1–4]. In a variety of cases, the polylogarithms' Hopf algebra structure [5–8] paved the way towards efficient manipulations and the recognition of the simplicity hidden in the resulting scattering amplitudes.

However, a growing list of iterated integrals from various field and string theories implies that multiple polylogarithms do not mark the end of the rope in terms of transcendental functions appearing in scattering amplitudes. For example, multiple polylogarithms fail to capture the two-loop sunset integral with non-zero masses [9–11], an eight-loop graph in ϕ^4 theory [12, 13] as well as the ten-point two-loop N^3MHV amplitude in $\mathcal{N} = 4$ super-Yang–Mills (sYM) theory [14]. The sunset integral and its generalization have recently been expressed in terms of elliptic di- and trilogarithms [10, 11, 15], whose connection to the language suggested below remains to be worked out. Considering in addition their appearance in one-loop open-string amplitudes, the situation calls for a systematic study and classification of the entire family of elliptic iterated integrals[1].

In the present article, we propose a framework for elliptic iterated integrals (or eIIs for short) and the associated periods, elliptic multiple zeta values (eMZVs). The framework aims at expressing scattering amplitudes in a variety of theories, and we here apply the techniques to one-loop amplitudes in open string theory as a first example. The language employed in the present article is primarily inspired by refs. [16,17], while refs. [18–22] contain further information on the mathematical background.

As opposed to multiple polylogarithms, which can be defined using just one type of differential form, elliptic iterated integrals require an infinite tower thereof [16]. These differential forms are based on a certain non-holomorphic extension of a classical Eisenstein-Kronecker series [23, 16], and we show how they can be used to naturally characterize and label elliptic iterated integrals as well as eMZVs. We investigate their relations, which results in constructive algorithms to perform amplitude computations.

In the same way as multiple zeta values (MZVs) arise from multiple polylogarithms at unit argument, the evaluation of iterated integrals along a certain path of an elliptic curve leads to structurally interesting periods, the eMZVs [17] mentioned above. These are certain analogues of the standard MZVs, which are related to elliptic associators [24] in the same way as MZVs are related to the Drinfeld associator [25–27]. However, the precise connection is beyond the scope of the current article. Given their ubiquitous appearance in the subsequent string amplitude computation, we will investigate eMZVs and discuss some of their properties as well as their $\mathbb{Q}$-linear relations.

The description of string scattering amplitudes via punctured Riemann surfaces at various genera directly leads to iterated integrals at the corresponding loop order. In particular, the disk integrals in open-string[2] tree-level amplitudes closely resemble multiple polylogarithms. Initially addressed via hypergeometric functions in refs. [34, 35], the α'-expansion of disk amplitudes finally proved to be a rich laboratory for MZVs. Their pattern of appearance has been understood

[1] The elliptic iterated integrals discussed in this work shall not be confused with elliptic integrals determining the arc length of an ellipse.

[2] In comparison to open-string amplitudes at tree-level, MZVs occurring in closed-string tree amplitudes [28,29] are constrained by the single-valued projection, see [30, 31] for mathematics and [32, 33] for physics literature.

in terms of mathematical structures such as motivic MZVs [7, 29] and the Drinfeld associator [36–38]. Explicit expressions with any number of open-string states can be determined using polylogarithm manipulations [3] or a matrix representation of the associator [38]. A variety of examples are available for download at the website [39].

The calculation of one-loop open-string amplitudes involves worldsheet integrals of cylinder and Möbius-strip topology [40]. In the current article, we focus on iterated integrals over a single cylinder boundary and leave the other topologies for later. Recognizing the cylinder as a genus-one surface with boundaries, it is not surprising that the α'-expansion of one-loop open-string amplitudes is a natural, simple and representative framework for the application of eIIs and eMZVs. We will explicitly perform calculations at four and five points for low orders in α' in order to demonstrate their usefulness. Higher multiplicities and orders in α' are argued to yield eMZVs and Eisenstein series on general grounds. In summary, one-loop string amplitudes turn out to be an ideal testing ground for the study of eMZVs, in particular because they appear in a more digestible context as compared to their instances in field theory.

This article is organized as follows: In section 2, we start by reviewing multiple polylogarithms and show, how their structure suggests a generalization to genus one. The appropriate differential forms and doubly-periodic functions are discussed and put into a larger mathematical context in section 3. Section 4 is devoted to the application of eIIs and eMZVs to the four-point one-loop amplitude of the open string, while section 5 contains a discussion of its multi-particle generalization.

2 Iterated integrals on an elliptic curve

After recalling the definition of multiple polylogarithms as well as several conventions, we will introduce elliptic iterated integrals (eIIs) as their genus-one analogues. While we will limit ourselves to basic definitions and calculational tools in the current section, a thorough introduction to the mathematical background of doubly-periodic functions will be provided in section 3.

2.1 Multiple polylogarithms

Multiple polylogarithms are defined by[3]

$$G(a_1, a_2, \ldots, a_n; z) \equiv \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t) \tag{2.1}$$

where $G(;z) \equiv 1$ apart from $G(\vec{a}; 0) = G(;0) = 0$. Below, we will refer to $\vec{a} = (a_1, \ldots, a_n)$ as the *label* and call z the *argument* of the polylogarithm G. Powers of ordinary logarithms can be conveniently represented in terms of multiple polylogarithms via

$$G(\underbrace{0, 0, \ldots, 0}_{n}; z) = \frac{1}{n!} \ln^n z, \qquad G(\underbrace{1, 1 \ldots, 1}_{n}; z) = \frac{1}{n!} \ln^n (1-z) \quad \text{and}$$
$$G(\underbrace{a, a, \ldots, a}_{n}; z) = \frac{1}{n!} \ln^n \left(1 - \frac{z}{a}\right) . \tag{2.2}$$

In addition, multiple polylogarithms satisfy the scaling property

$$G(ka_1, ka_2, \ldots, ka_n; kz) = G(a_1, a_2, \ldots, a_n; z) , \qquad k \neq 0 , \quad a_n \neq 0 , \quad z \neq 0 , \tag{2.3}$$

[3]The conventions for multiple polylogarithms used in this paper agree with those in refs. [5, 29, 41]. Other aspects of multiple polylogarithms are discussed for example in references [42, 43].

whose interplay with a general shuffle regularization will be discussed below eq. (2.9). Another property is referred to as the Hölder convolution [44]: for $a_1 \neq 1$ and $a_n \neq 0$ one finds

$$G(a_1,\ldots,a_n;1) = \sum_{k=0}^{n} (-1)^k\, G\Big(1-a_k,\ldots,1-a_1;1-\frac{1}{p}\Big)\, G\Big(a_{k+1},\ldots,a_n;\frac{1}{p}\Big) \tag{2.4}$$

for all $p \in \mathbb{C}\setminus\{0\}$. Multiple polylogarithms constitute a graded commutative algebra with the shuffle product [5–8]

$$\begin{aligned} G(a_1,\ldots,a_r;z)G(a_{r+1},\ldots,a_{r+s};z) &= \sum_{\sigma\in\Sigma(r,s)} G(a_{\sigma(1)},\ldots,a_{\sigma(r+s)};z) \\ &\equiv G\big((a_1,\ldots,a_r) ⧢ (a_{r+1},\ldots,a_{r+s});z\big)\ , \end{aligned} \tag{2.5}$$

where the shuffle $\Sigma(r,s)$ is the subset of the permutation group S_{r+s} acting on $\{a_1,\ldots,a_{r+s}\}$ which leaves the order of the elements of the individual tuples $\{a_1,\ldots,a_r\}$ and $\{a_{r+1},\ldots,a_{r+s}\}$ unchanged. The unit element for shuffling is $G(;z)$=1.

MZVs are special cases of multiple polylogarithms with labels $a_i \in \{0,1\}$ evaluated at argument $z=1$:

$$\zeta_{n_1,\ldots,n_r} = (-1)^r G(\underbrace{0,0,\ldots,0,1}_{n_r},\ldots,\underbrace{0,0,\ldots,0,1}_{n_1};1)\,, \tag{2.6}$$

where the numbers below the underbraces denote the number of entries[4].

From the definition (2.1) it is obvious that multiple polylogarithms diverge when either $a_1 = z$ or $a_n = 0$. As discussed in refs. [5, 6], the general idea for regularizing the integrals is to slightly move the endpoints of the integration by a small parameter and to afterwards expand in this parameter. The regularized value of the polylogarithm is defined to be the piece independent of the regularization parameter, which can be extracted using shuffle relations. For the case where $a_1 = z$ the regularized value can be obtained via

$$\begin{aligned} G(z,a_2,\ldots,a_n;z) &= G(z;z)\,G(a_2,\ldots,a_n;z) - G(a_2,z,a_3,\ldots,a_n;z) \\ &\quad - G(a_2,a_3,z,a_4,\ldots,a_n;z) - \ldots - G(a_2,\ldots,a_n,z;z) \end{aligned} \tag{2.7}$$

where one defines

$$G(z,\ldots,z;z) = 0\ . \tag{2.8}$$

The situation, where $a_n = 0$ can be dealt with accordingly

$$\begin{aligned} G(a_1,a_2,\ldots,a_{n-1},0;z) &= G(a_1,a_2,\ldots,a_{n-1};z)\,G(0;z) - G(a_1,a_2,\ldots,0,a_{n-1};z) \\ &\quad - G(a_1,a_2,\ldots,0,a_{n-2},a_{n-1};z) - \ldots - G(0,a_2,\ldots,a_{n-1};z)\,, \end{aligned} \tag{2.9}$$

where now, however, $G(0;z) = \ln(z) \neq 0$. Although the above rewriting keeps the pure logarithms explicit, it will nevertheless prove convenient in order to bypass subtleties of the identity eq. (2.11) below. Multiple polylogarithms are understood to be shuffle-regularized in a way compatible with eq. (2.3).

Regularization of multiple polylogarithms can be straightforwardly translated to MZVs. All MZVs $\zeta_{n_1,\ldots,n_r}$ with $n_r = 1$ are defined by their shuffled version eq. (2.7). Employing eq. (2.3), one finds $G(1,\ldots,1;1) = 0$ from eq. (2.8) immediately.

[4]Our convention for MZVs agrees with refs. [5, 29, 45].

2.1.1 Removing the argument z from the label

Starting from an arbitrary iterated integral, the corresponding polylogarithm can not always be determined straightforwardly: whenever the argument appears in the label $\vec{a}$, an integration using eq. (2.1) is impossible. Solving this problem requires a rewriting of the multiple polylogarithm

$$G(\{0, a_1, a_2, \ldots, a_n, z\}; z) \tag{2.10}$$

in terms of polylogarithms whose labels are free of the argument. In the above equation $\{a, b, \ldots\}$ refers to a word built from the letters $a, b, \ldots$. Polylogarithms of the special form $G(\vec{a}, z)$ with $a_i \in \{0, z\}$ can be rescaled to yield MZVs using eq. (2.3) provided that the last entry of $\vec{a}$ is different from zero. In a generic situation, the relation [3]

$$\begin{aligned}
G(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n; z) = {} & G(a_{i-1}, a_1, \ldots, a_{i-1}, \hat{z}, a_{i+1}, \ldots, a_n; z) && \text{(2.11a)}\\
& - G(a_{i+1}, a_1, \ldots, a_{i-1}, \hat{z}, a_{i+1}, \ldots, a_n; z) && \text{(2.11b)}\\
& - \int_0^z \frac{\mathrm{d}t}{t - a_{i-1}} G(a_1, \ldots, \hat{a}_{i-1}, t, a_{i+1}, \ldots, a_n; t) && \text{(2.11c)}\\
& + \int_0^z \frac{\mathrm{d}t}{t - a_{i+1}} G(a_1, \ldots, a_{i-1}, t, \hat{a}_{i+1}, \ldots, a_n; t) && \text{(2.11d)}\\
& + \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n; t) && \text{(2.11e)}
\end{aligned}$$

allows to recursively remove the argument z from the labels of a multiple polylogarithm, because the expressions on the right-hand side either have shorter labels or are free of z. A hat denotes the omission of the respective label, and it is assumed that at least one $a_j \neq 0$. The availability of a recursive formula like eq. (2.11) is intrinsic to the moduli space of Riemann spheres with marked points [46]. An explicit discussion including algorithms is ref. [47].

As an identity similar to eq. (2.11) will be crucial in deriving relations for eIIs in subsection 2.2 below, let us briefly comment on the application and generalization of eq. (2.11): If the argument z appears multiple times in the label $\vec{a}$, the first four terms on the right hand side (terms (2.11a) to (2.11d)) have to be evaluated for each occurrence of z. The reduction will lead to expressions where the labels of the polylogarithms on the right hand side are independent of z or shorter, which is ensured by cancellations between neighboring terms. If $a_n = z$, the term (2.11d) has to be dropped and the term (2.11b) needs to be altered to $-G(0, a_1, \ldots, a_{i-1}, \hat{z}; z)$.

Multiple polylogarithms with $a_1 = z$ require special attention as well. However, in order to keep the exposition simple, we will assume that those polylogarithms have already been taken care of by applying the shuffle regularization rule eq. (2.7).

The following examples (with $a_j \neq z$) are typical relations derived from the above identity:

$$\begin{aligned}
G(a_1, 0, z; z) &= G(0, 0, a_1; z) - G(0, a_1, a_1; z) - G(a_1; z)\zeta_2\\
G(a_1, z, a_2; z) &= G(a_1, a_1, a_2; z) - G(a_2, 0, a_1; z) + G(a_2, a_1, a_1; z) - G(a_2, a_1, a_2; z)\,.
\end{aligned} \tag{2.12}$$

Proving eq. (2.11) is straightforward. It relies on writing the polylogarithm on the left hand side as the integral over its total derivative and using partial fraction as well as relations (A.1) to (A.3) in appendix A. Finally, let us note that eq. (2.11) preserves shuffle regularization. The complete proof of eq. (2.11) as well as numerous examples are contained in section 5 of ref. [3]. A collection of identities between MZVs can be found in the multiple zeta value data mine [48].

2.2 Iterated integrals on an elliptic curve

In this subsection we are going to take a first look at eIIs. In the following exposition, we will omit several mathematical details, which will be discussed in section 3 below. As eIIs will turn out to be a generalization of the multiple polylogarithms discussed above, we will follow the structure of the previous subsection closely.

In eq. (2.1), the differential dt is weighted by

$$\frac{1}{t-a_i}\,, \tag{2.13}$$

which yields iterated integrals on the genus-zero curve $\mathbb{C}\setminus\{a_1,...,a_n\}$. Here, we propose a generalization to eIIs. An infinite number of weighting functions $f^{(n)}$ of weights $n=0,1,2,\dots$ is necessary, whose appearance will be justified and whose precise definition will be provided in section 3. They lead to eIIs in the same way as does eq. (2.13) at genus zero. Accordingly, the functions $f^{(n)}(z,\tau)$ are doubly periodic with respect to the two cycles of the torus, with modular parameter τ in the upper half plane

$$f^{(n)}(z,\tau)=f^{(n)}(z+1,\tau)\quad\text{and}\quad f^{(n)}(z,\tau)=f^{(n)}(z+\tau,\tau)\ . \tag{2.14}$$

Below, we are going to suppress the τ-dependence and will simply write $f^{(n)}(z)$. As will be explained in subsection 3.3, the functions $f^{(n)}$ are known for all non-negative integer weights n. In particular they are non-holomorphic and expressible in terms of the odd Jacobi function $\theta_1(z,\tau)$, e.g.

$$f^{(0)}(z)\equiv 1\ ,\qquad f^{(1)}(z)\equiv\partial\ln\theta_1(z,\tau)+2\pi i\frac{\operatorname{Im}z}{\operatorname{Im}\tau} \tag{2.15}$$

$$f^{(2)}(z)\equiv\frac{1}{2}\Big[\Big(\partial\ln\theta_1(z,\tau)+2\pi i\frac{\operatorname{Im}z}{\operatorname{Im}\tau}\Big)^2+\partial^2\ln\theta_1(z,\tau)-\frac{1}{3}\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}\Big] \tag{2.16}$$

where ∂ and $'$ denote a derivative in the first argument of θ_1. Their parity alternates depending on the weight n:

$$f^{(n)}(-z)=(-1)^n f^{(n)}(z)\,. \tag{2.17}$$

The functions $f^{(n)}$ are defined for arbitrary complex arguments z. Restricting to real arguments z, however, will not only simplify eqs. (2.15) and (2.16) but in addition lead to the system of iterated integrals appropriate for the one-loop open-string calculations in sections 4 and 5 below. Hence, in the remainder of the current section, any argument and label of the eIIs to be defined is assumed to be real. We will comment on the additional ingredients required for generic complex arguments z and relate them to multiple elliptic polylogarithms in subsection 3.1.

Employing the functions $f^{(n)}$, eIIs are defined in analogy to eq. (2.1) via

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r\end{smallmatrix};z\right)\equiv\int_0^z \mathrm{d}t\, f^{(n_1)}(t-a_1)\ \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r\end{smallmatrix};t\right), \tag{2.18}$$

where the recursion starts with $\Gamma(;z)\equiv 1$. Following the terminology used for $f^{(n)}$ above, the eII in eq. (2.18) is said to have *weight* $\sum_{i=1}^r n_i$, and the number r of integrations will be referred to as its *length*.

The definition of eIIs directly implies a shuffle relation with respect to the combined letters

$A_i \equiv {}^{n_i}_{a_i}$ describing the integration weights $f^{(n_i)}(z-a_i)$,

$$\Gamma(A_1,A_2,\ldots,A_r;z)\,\Gamma(B_1,B_2,\ldots,B_q;z)=\Gamma\big((A_1,A_2,\ldots,A_r)\sqcup\!\sqcup(B_1,B_2,\ldots,B_q);z\big)\,, \quad (2.19)$$

where the shuffle symbol has been defined in eq. (2.5). Another immediate consequence of definition (2.18) is the reflection identity

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r\end{smallmatrix};z\right)=(-1)^{n_1+n_2+\ldots+n_r}\,\Gamma\left(\begin{smallmatrix} n_r & \ldots & n_2 & n_1 \\ z-a_r & \ldots & z-a_2 & z-a_1\end{smallmatrix};z\right)\,. \quad (2.20)$$

Formally reminiscent of the Hölder convolution in eq. (2.4), the above reflection identity is valid for all arguments $z\in\mathbb{C}\setminus\{0\}$. It can be proven using the parity properties of the weighting functions $f^{(n)}$ in eq. (2.17) and a reparametrization of the integration domain. If all the labels a_i vanish, we will often use the notation

$$\Gamma(n_1,n_2,\ldots,n_r;z)\equiv\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ 0 & 0 & \ldots & 0\end{smallmatrix};z\right)\,. \quad (2.21)$$

2.2.1 Elliptic multiple zeta values

Evaluating eIIs with all a_i equal to 0 (or equivalently $a_i=1$ by the periodicity property eq. (2.14)) at $z=1$ gives rise to iterated integrals

$$\begin{aligned}\omega(n_1,n_2,\ldots,n_r)&\equiv\int\limits_{0\le z_i\le z_{i+1}\le 1} f^{(n_1)}(z_1)\mathrm{d}z_1\,f^{(n_2)}(z_2)\mathrm{d}z_2\,\ldots f^{(n_r)}(z_r)\mathrm{d}z_r \quad (2.22)\\ &=\Gamma(n_r,\ldots,n_2,n_1;1)\end{aligned}$$

which we will refer to as *elliptic multiple zeta values* or *eMZV*s for short. They furnish a natural genus-one generalization of standard MZVs[5] as defined in eq. (2.6). The shuffle relation eq. (2.19) can be straightforwardly applied to eMZVs

$$\omega(n_1,n_2,\ldots,n_r)\omega(k_1,k_2,\ldots,k_s)=\omega\big((n_1,n_2,\ldots,n_r)\sqcup\!\sqcup(k_1,k_2,\ldots,k_s)\big)\,, \quad (2.23)$$

and the parity property eq. (2.17) of the functions $f^{(n)}$ implies the reflection identity

$$\omega(n_1,n_2,\ldots,n_{r-1},n_r)=(-1)^{n_1+n_2+\ldots+n_r}\omega(n_r,n_{r-1},\ldots,n_2,n_1)\,. \quad (2.24)$$

Note that a similar set of ω's can be defined by an iterated integral along the path from 0 to τ replacing the integration domain $[0,1]$ in eq. (2.22). They appear in the modular transformations of eMZVs and naturally satisfy the properties eqs. (2.23) and (2.24) as well. Likewise, the eIIs defined in eq. (2.18) allow for a version with integrations on the path from 0 to τ.

Regularization. Among the family of functions $f^{(n)}(z)$ used to define eIIs and eMZVs, only $f^{(1)}(z)$ has a simple pole at zero and its images under the translations in eq. (2.14). Therefore, iterated integrals of the form

$$\Gamma\left(\begin{smallmatrix} n_r & \ldots & n_2 & n_1 \\ a_r & \ldots & a_2 & a_1\end{smallmatrix};z\right)=\int\limits_{0\le z_i\le z_{i+1}\le z} f^{(n_1)}(z_1-a_1)\mathrm{d}z_1\,f^{(n_2)}(z_2-a_2)\mathrm{d}z_2\,\ldots f^{(n_r)}(z_r-a_r)\mathrm{d}z_r \quad (2.25)$$

[5]In order to distinguish between eMZVs and MZVs, we will sometimes refer to the latter as *standard* MZVs.

with $n_1 = 1$ or $n_r = 1$ need to be regularized if either $a_1 = 0$ or $a_r = z$. As with multiple polylogarithms, the idea is to slightly move the endpoints of the integration domain by a small parameter, and then to expand in this parameter. More precisely, one writes the integral

$$\int\limits_{\varepsilon \leq z_i \leq z_{i+1} \leq z-\varepsilon} f^{(n_1)}(z_1 - a_1)\mathrm{d}z_1\, f^{(n_2)}(z_2 - a_2)\mathrm{d}z_2 \dots f^{(n_r)}(z_r - a_r)\mathrm{d}z_r \tag{2.26}$$

as a polynomial in $\ln(-2\pi i\varepsilon)$, where the branch of the logarithm is chosen such that we have $\ln(-i) = -\frac{\pi i}{2}$. The regularized value of eq. (2.26) is then defined to be the constant term in this expansion. The additional $-2\pi i$ in the expansion parameter $\ln(-2\pi i\varepsilon)$ ensures that no logarithms appear in the limit $\tau \to i\infty$, and that eMZVs degenerate to MZVs. A thorough treatment of this degeneration can be found in ref. [24] and will be exploited in ref. [49].

2.2.2 Removing the argument z from the label

As for the multiple polylogarithms, no arguments z are allowed in the labels $\{a_1 \dots a_r\}$ in order to perform the integration using eq. (2.18). Therefore we need to find relations, which trade eIIs with one or multiple occurrences of the argument z in the label for eIIs where z appears in the argument exclusively. The key idea for finding those relations is to write the eII as the integral of its total derivative

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_q & \dots & n_r \\ a_1 & a_2 & \dots & z & \dots & a_r \end{smallmatrix}; z\right) = \int_0^z \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_q & \dots & n_r \\ a_1 & a_2 & \dots & t & \dots & a_r \end{smallmatrix}; t\right) + \lim_{z\to 0} \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_q & \dots & n_r \\ a_1 & a_2 & \dots & z & \dots & a_r \end{smallmatrix}; z\right). \tag{2.27}$$

This resembles the strategy at genus zero which led to the identity eq. (2.11) between multiple polylogarithms. In the subsequent, we address additional features and subtleties intrinsic to the elliptic case. The feasibility of this approach in the elliptic scenario is discussed in ref. [16], see in particular theorem 26 therein.

Boundary terms. The boundary term at $z = 0$ usually drops out from eq. (2.27) due to the vanishing volume of the integration domain. However, the special situation when all $n_j = 1$ leads to the appearance of standard MZVs. As will be elaborated on in section 3, the function $f^{(1)}$ is the only source of singularities in the integration variables, as can be seen from its leading behavior $f^{(1)}(z) = z^{-1} + \mathcal{O}(z)$. Hence, the regime $z \to 0$ reproduces multiple polylogarithms as defined in eq. (2.1):

$$\begin{aligned} \lim_{z\to 0} \Gamma\left(\begin{smallmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_r \end{smallmatrix}; z\right) &= \lim_{z\to 0} \int_0^z \frac{\mathrm{d}t_1}{t_1 - a_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2 - a_2} \dots \int_0^{t_{r-1}} \frac{\mathrm{d}t_r}{t_r - a_r} \\ &= \lim_{z\to 0} G(a_1, a_2, \dots, a_r; z). \end{aligned} \tag{2.28}$$

If all $a_j \in \{0, z\}$, the scaling relation eq. (2.3) allows to rewrite the polylogarithms in terms of MZVs (see eq. (2.6)), leading to

$$\lim_{z\to 0} \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ b_1 z & b_2 z & \dots & b_r z \end{smallmatrix}; z\right) = G(b_1, b_2, \dots, b_r; 1) \prod_{j=1}^{r} \delta_{n_j,1}, \quad b_j \in \{0, 1\}. \tag{2.29}$$

Partial derivatives. The total t-derivative in eq. (2.27) can be written in terms of partial derivatives with respect to the arguments and the labels. This requires the elliptic analogues of eqns. (A.1) to (A.3) listed below in order to arrive at shorter elliptic polylogarithms. The derivative with respect to the argument

$$\frac{\partial}{\partial z}\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix};z\right) = f^{(n_1)}(z-a_1)\,\Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix};z\right) \tag{2.30}$$

follows straightforwardly from eq. (2.18). Slightly more work using $\frac{\partial}{\partial a}f^{(n)}(t-a) = -\frac{\partial}{\partial t}f^{(n)}(t-a)$ as well as eq. (2.30) is required for derivatives with respect to labels a_q. Starting with the special cases $q=1$ and $q=r$ one finds

$$\begin{aligned}
\frac{\partial}{\partial a_1}\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix};t_0\right) &= -f^{(n_1)}(t_0-a_1)\,\Gamma\left(\begin{smallmatrix} n_2 & n_3 & \dots & n_r \\ a_2 & a_3 & \dots & a_r \end{smallmatrix};t_0\right) \\
&\quad + \int_0^{t_0} \mathrm{d}t\, f^{(n_1)}(t-a_1)f^{(n_2)}(t-a_2)\,\Gamma\left(\begin{smallmatrix} n_3 & \dots & n_r \\ a_3 & \dots & a_r \end{smallmatrix};t\right) \qquad (2.31) \\
\frac{\partial}{\partial a_r}\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix};t_0\right) &= f^{(n_r)}(-a_r)\,\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_{r-1} \\ a_1 & a_2 & \dots & a_{r-1} \end{smallmatrix};t_0\right) \\
&\quad - \left(\prod_{j=1}^{r-2}\int_0^{t_{j-1}} \mathrm{d}t_j\, f^{(n_j)}(t_j-a_j)\right)\int_0^{t_{r-2}} \mathrm{d}t\, f^{(n_{r-1})}(t-a_{r-1})f^{(n_r)}(t-a_r)\,. \qquad (2.32)
\end{aligned}$$

Deriving with respect to a label a_q with $q \neq 1, r$ yields

$$\begin{aligned}
&\frac{\partial}{\partial a_q}\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix};t_0\right) \\
&= \left(\prod_{j=1}^{q-1}\int_0^{t_{j-1}} \mathrm{d}t_j\, f^{(n_j)}(t_j-a_j)\right)\int_0^{t_{q-1}} \mathrm{d}t\, f^{(n_q)}(t-a_q)f^{(n_{q+1})}(t-a_{q+1})\,\Gamma\left(\begin{smallmatrix} n_{q+2} & \dots & n_r \\ a_{q+2} & \dots & a_r \end{smallmatrix};t\right) \\
&- \left(\prod_{j=1}^{q-2}\int_0^{t_{j-1}} \mathrm{d}t_j\, f^{(n_j)}(t_j-a_j)\right)\int_0^{t_{q-2}} \mathrm{d}t\, f^{(n_{q-1})}(t-a_{q-1})f^{(n_q)}(t-a_q)\,\Gamma\left(\begin{smallmatrix} n_{q+1} & \dots & n_r \\ a_{q+1} & \dots & a_r \end{smallmatrix};t\right). \qquad (2.33)
\end{aligned}$$

Total derivatives. Summing the above partial derivatives with respect to the argument z and the labels a_q, total derivatives from eq. (2.27) can be expressed in a very efficient way. For a single instance of $a_q = z$, the special cases $q=1$ and $q=r$ give rise to

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ t_0 & a_2 & \dots & a_r \end{smallmatrix};t_0\right) = \int_0^{t_0} \mathrm{d}t\, f^{(n_1)}(t-t_0)f^{(n_2)}(t-a_2)\,\Gamma\left(\begin{smallmatrix} n_3 & \dots & n_r \\ a_3 & \dots & a_r \end{smallmatrix};t\right) \quad \text{and} \tag{2.34}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t_0}\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{r-1} & n_r \\ a_1 & \dots & a_{r-1} & t_0 \end{smallmatrix};t_0\right) &= f^{(n_1)}(t_0-a_1)\,\Gamma\left(\begin{smallmatrix} n_2 & \dots & n_{r-1} & n_r \\ a_2 & \dots & a_{r-1} & t_0 \end{smallmatrix};t_0\right) + f^{(n_r)}(-t_0)\,\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{r-1} \\ a_1 & \dots & a_{r-1} \end{smallmatrix};t_0\right) \\
&\quad -\left(\prod_{j=1}^{r-2}\int_0^{t_{j-1}} \mathrm{d}t_j\; f^{(n_j)}(t_j-a_j)\right)\int_0^{t_{r-2}} \mathrm{d}t\, f^{(n_{r-1})}(t-a_{r-1})f^{(n_r)}(t-t_0)\;. \qquad (2.35)
\end{aligned}$$

For $q \neq 1, r$, the integrand of eq. (2.27) takes the form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t_0}\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_{q-1} & n_q & n_{q+1} & \dots & n_r \\ a_1 & a_2 & \dots & a_{q-1} & t_0 & a_{q+1} & \dots & a_r \end{smallmatrix};t_0\right) = f^{(n_1)}(t_0-a_1)\,\Gamma\left(\begin{smallmatrix} n_2 & \dots & n_{q-1} & n_q & n_{q+1} & \dots & n_r \\ a_2 & \dots & a_{q-1} & t_0 & a_{q+1} & \dots & a_r \end{smallmatrix};t_0\right) \\
&+ \left(\prod_{j=1}^{q-1}\int_0^{t_{j-1}} \mathrm{d}t_j\, f^{(n_j)}(t_j-a_j)\right)\int_0^{t_{q-1}} \mathrm{d}t\, f^{(n_q)}(t-t_0)f^{(n_{q+1})}(t-a_{q+1})\,\Gamma\left(\begin{smallmatrix} n_{q+2} & \dots & n_r \\ a_{q+2} & \dots & a_r \end{smallmatrix};t\right) \qquad (2.36) \\
&- \left(\prod_{j=1}^{q-2}\int_0^{t_{j-1}} \mathrm{d}t_j\, f^{(n_j)}(t_j-a_j)\right)\int_0^{t_{q-2}} \mathrm{d}t\, f^{(n_{q-1})}(t-a_{q-1})f^{(n_q)}(t-t_0)\,\Gamma\left(\begin{smallmatrix} n_{q+1} & \dots & n_r \\ a_{q+1} & \dots & a_r \end{smallmatrix};t\right).
\end{aligned}$$

Further examples with repeated appearances of t_0 are displayed in appendix B.1.

Fay identities. Having applied the above derivative identities, one is usually left with expressions containing integrals of the form

$$\int_0^z \mathrm{d}t \; f^{(n_1)}(t-a_1) f^{(n_2)}(t-a_2) \, , \tag{2.37}$$

where the integration variable appears in the argument of more than one function $f^{(n)}$. In the corresponding situation for multiple polylogarithms, with weights of the form eq. (2.13), one would have used partial fraction identities

$$\frac{1}{(t-a)(t-b)} = \frac{1}{(t-a)(a-b)} + \frac{1}{(t-b)(b-a)} \tag{2.38}$$

in order to avoid the repeated appearance of the integration variable t. Analogous relations for the more general class of weighting functions $f^{(n)}$ are provided by Fay identities, which will be put in a larger mathematical context in section 3 below. They relate products $f^{(n_1)} f^{(n_2)}$ at arguments x, t and $x-t$ and thereby allow to systematically remove repeated appearances of some integration variable. A simple example of a Fay identity relates products of functions $f^{(1)}$ to a sum of functions $f^{(2)}$

$$f^{(1)}(t-x) f^{(1)}(t) = f^{(1)}(t-x) f^{(1)}(x) - f^{(1)}(t) f^{(1)}(x) + f^{(2)}(t) + f^{(2)}(x) + f^{(2)}(t-x) \, . \tag{2.39}$$

The general relation, which is valid for complex arguments x, t as well,

$$\begin{aligned} f^{(n_1)}(t-x) f^{(n_2)}(t) = &-(-1)^{n_1} f^{(n_1+n_2)}(x) + \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} f^{(n_2-j)}(x) f^{(n_1+j)}(t-x) \\ &+ \sum_{j=0}^{n_1} \binom{n_2-1+j}{j} (-1)^{n_1+j} f^{(n_1-j)}(x) f^{(n_2+j)}(t) \, , \end{aligned} \tag{2.40}$$

in turn allows to remove all repeated occurrences of the variable t. Iterating the above steps, one can thus eliminate all arguments from the label of any eII recursively.

Result. Combining the Fay identity eq. (2.40) with the total derivatives in eqns. (2.34) to (2.36) turns (2.27) into a recursive rule for removing the argument z from the label of $\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_q & \dots & n_r \\ a_1 & \dots & z & \dots & a_r \end{smallmatrix} ; z \right)$. In the equations below, all terms on the right-hand side are either free of $a_q = z$ or have shorter labels. The special cases $q=1$ and $q=r$ yield

$$\begin{aligned} \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ z & a_2 & \dots & a_r \end{smallmatrix} ; z \right) = &\lim_{z \to 0} G(z, a_2, \dots, a_r; z) \prod_{j=1}^{r} \delta_{n_j,1} - (-1)^{n_1} \Gamma\left(\begin{smallmatrix} n_1+n_2 & 0 & n_3 & \dots & n_r \\ a_2 & 0 & a_3 & \dots & a_r \end{smallmatrix} ; z \right) \\ &+ \sum_{j=0}^{n_1} (-1)^{n_1+j} \binom{n_2-1+j}{j} \Gamma\left(\begin{smallmatrix} n_1-j & n_2+j & n_3 & \dots & n_r \\ a_2 & a_2 & a_3 & \dots & a_r \end{smallmatrix} ; z \right) \\ &+ \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} \int_0^z \mathrm{d}t \, f^{(n_2-j)}(t-a_2) \, \Gamma\left(\begin{smallmatrix} n_1+j & n_3 & \dots & n_r \\ t & a_3 & \dots & a_r \end{smallmatrix} ; t \right) \end{aligned} \tag{2.41}$$

$$\begin{aligned} \Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{r-1} & n_r \\ a_1 & \dots & a_{r-1} & z \end{smallmatrix} ; z \right) = &\lim_{z \to 0} G(a_1, \dots, a_{r-1}, z; z) \prod_{j=1}^{r} \delta_{n_j,1} + \int_0^z \mathrm{d}t \, f^{(n_1)}(t-a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_{r-1} & n_r \\ a_2 & \dots & a_{r-1} & t \end{smallmatrix} ; t \right) \\ &+ (-1)^{n_r} \Gamma\left(\begin{smallmatrix} n_r & n_1 & \dots & n_{r-1} \\ 0 & a_1 & \dots & a_{r-1} \end{smallmatrix} ; z \right) + (-1)^{n_r} \Gamma\left(\begin{smallmatrix} n_{r-1}+n_r & n_1 & \dots & n_{r-2} & 0 \\ a_{r-1} & a_1 & \dots & a_{r-2} & 0 \end{smallmatrix} ; z \right) \end{aligned}$$

$$- \sum_{j=0}^{n_{r-1}} \binom{n_r - 1 + j}{j} \int_0^z \mathrm{d}t \, f^{(n_{r-1}-j)}(t - a_{r-1}) \, \Gamma \left(\begin{smallmatrix} n_1 & \dots & n_{r-2} & n_r+j \\ a_1 & \dots & a_{r-2} & t \end{smallmatrix} ; t \right)$$
$$- \sum_{j=0}^{n_r} \binom{n_{r-1} - 1 + j}{j} (-1)^{n_r+j} \, \Gamma \left(\begin{smallmatrix} n_r-j & n_1 & \dots & n_{r-2} & n_{r-1}+j \\ a_{r-1} & a_1 & \dots & a_{r-2} & a_{r-1} \end{smallmatrix} ; z \right) , \tag{2.42}$$

while $a_q = z$ at a generic position $q \neq 1, r$ can be addressed via

$$\begin{aligned} \Gamma \left(\begin{smallmatrix} n_1 & n_2 & \dots & n_{q-1} & n_q & n_{q+1} & \dots & n_r \\ a_1 & a_2 & \dots & a_{q-1} & z & a_{q+1} & \dots & a_r \end{smallmatrix} ; z \right) &= \lim_{z \to 0} G(a_1, \dots, a_{q-1}, z, a_{q+1}, \dots, a_r; z) \prod_{j=1}^r \delta_{n_j,1} \\ &+ \int_0^z \mathrm{d}t \, f^{(n_1)}(t - a_1) \, \Gamma \left(\begin{smallmatrix} n_2 & \dots & n_{q-1} & n_q & n_{q+1} & \dots n_r \\ a_2 & \dots & a_{q-1} & t & a_{q+1} & \dots a_r \end{smallmatrix} ; t \right) \\ &- (-1)^{n_q} \, \Gamma \left(\begin{smallmatrix} n_q+n_{q+1} & n_1 & \dots & n_{q-1} & 0 & n_{q+2} & \dots & n_r \\ a_{q+1} & a_1 & \dots & a_{q-1} & 0 & a_{q+2} & \dots & a_r \end{smallmatrix} ; z \right) + (-1)^{n_q} \, \Gamma \left(\begin{smallmatrix} n_q+n_{q-1} & n_1 & \dots & n_{q-2} & 0 & n_{q+1} & \dots & n_r \\ a_{q-1} & a_1 & \dots & a_{q-2} & 0 & a_{q+1} & \dots & a_r \end{smallmatrix} ; z \right) \\ &+ \sum_{j=0}^{n_{q+1}} \binom{n_q - 1 + j}{j} \int_0^z \mathrm{d}t \, f^{(n_{q+1}-j)}(t - a_{q+1}) \, \Gamma \left(\begin{smallmatrix} n_1 & \dots & n_{q-1} & n_q+j & n_{q+2} & \dots & n_r \\ a_1 & \dots & a_{q-1} & t & a_{q+2} & \dots & a_r \end{smallmatrix} ; t \right) \\ &+ \sum_{j=0}^{n_q} \binom{n_{q+1} - 1 + j}{j} (-1)^{n_q+j} \, \Gamma \left(\begin{smallmatrix} n_q-j & n_1 & \dots & n_{q-1} & n_{q+1}+j & n_{q+2} & \dots & n_r \\ a_{q+1} & a_1 & \dots & a_{q-1} & a_{q+1} & a_{q+2} & \dots & a_r \end{smallmatrix} ; z \right) \\ &- \sum_{j=0}^{n_{q-1}} \binom{n_q - 1 + j}{j} \int_0^z \mathrm{d}t \, f^{(n_{q-1}-j)}(t - a_{q-1}) \, \Gamma \left(\begin{smallmatrix} n_1 & \dots & n_{q-2} & n_q+j & n_{q+1} & \dots & n_r \\ a_1 & \dots & a_{q-2} & t & a_{q+1} & \dots & a_r \end{smallmatrix} ; t \right) \\ &- \sum_{j=0}^{n_q} \binom{n_{q-1} - 1 + j}{j} (-1)^{n_q+j} \, \Gamma \left(\begin{smallmatrix} n_q-j & n_1 & \dots & n_{q-2} & n_{q-1}+j & n_{q+1} & \dots & n_r \\ a_{q-1} & a_1 & \dots & a_{q-2} & a_{q-1} & a_{q+1} & \dots & a_r \end{smallmatrix} ; z \right) . \end{aligned} \tag{2.43}$$

Situations with multiple successive appearance of $a_j = z$ are discussed in appendix B.

Examples. At length one, the reflection identity eq. (2.20) implies that

$$\Gamma \left(\begin{smallmatrix} n \\ z \end{smallmatrix} ; z \right) = (-1)^n \, \Gamma(n; z) , \tag{2.44}$$

which covers all identities at this length. At length two, cases with $n_1 = 0$ or $n_2 = 0$ are similarly determined by eq. (2.20), so the simplest non-trivial application of eq. (2.27) is $\Gamma \left(\begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix} ; z \right)$. The differential can be derived via eq. (2.31) and simplified using the Fay identity eq. (2.39) as well as eq. (2.44),

$$\frac{\mathrm{d}}{\mathrm{d}t} \Gamma \left(\begin{smallmatrix} 1 & 1 \\ t & 0 \end{smallmatrix} ; t \right) = 2 \, \Gamma(2; t) + f^{(2)}(t) \, \Gamma(0; t) - 2 f^{(1)}(t) \, \Gamma(1; t) , \tag{2.45}$$

see eq. (2.21) for the notation on the right hand side. In combination with the boundary term

$$\lim_{z \to 0} \Gamma \left(\begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix} ; z \right) = G(1, 0; 1) = \zeta_2 , \tag{2.46}$$

we find

$$\Gamma \left(\begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix} ; z \right) = 2 \, \Gamma(0, 2; z) + \Gamma(2, 0; z) - 2 \, \Gamma(1, 1; z) + \zeta_2 , \tag{2.47}$$

which of course agrees with the general formula eq. (2.41). The same reasoning can be applied recursively to obtain for example

$$\begin{aligned} \Gamma \left(\begin{smallmatrix} 1 & 1 & 1 \\ z & 0 & 0 \end{smallmatrix} ; z \right) = - \Gamma \left(\begin{smallmatrix} 1 & 1 & 1 \\ z & z & 0 \end{smallmatrix} ; z \right) &= - \Gamma(0, 3, 0; z) - \Gamma(0, 0, 3; z) - 3 \, \Gamma(1, 1, 1; z) + \Gamma(2, 0, 1; z) \\ &+ \Gamma(1, 2, 0; z) + 2 \, \Gamma(0, 2, 1; z) + 2 \, \Gamma(1, 0, 2; z) + \zeta_2 \, \Gamma(1; z) - \zeta_3 \end{aligned} \tag{2.48}$$

$$\Gamma \left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ z & 0 & 0 & 0 \end{smallmatrix} ; z \right) = 2 \, \Gamma(0, 0, 0, 2; z) + \Gamma(0, 0, 2, 0; z) - 2 \, \Gamma(0, 0, 1, 1; z) + \zeta_2 \, \Gamma(0, 0; z) \tag{2.49}$$

as well as

$$\Gamma\left(\begin{smallmatrix}0&1&0&1&0\\0&z&0&0&0\end{smallmatrix};z\right)=2\,\Gamma(0,0,0,2,0;z)+\Gamma(0,2,0,0,0;z)-2\,\Gamma(0,1,0,1,0;z) \tag{2.50}$$

$$\begin{aligned}\Gamma\left(\begin{smallmatrix}0&1&1&0&0\\0&z&0&0&0\end{smallmatrix};z\right)&=\Gamma(0,0,2,0,0;z)+\Gamma(0,0,0,2,0;z)+\Gamma(2,0,0,0,0;z)\\&\quad-\Gamma(1,0,1,0,0;z)-\Gamma(1,0,0,1,0;z)\ .\end{aligned} \tag{2.51}$$

In subsection 4.3 these relations turn out to be crucial to express the low energy expansion of one-loop string amplitudes in terms of eMZVs.

The most general relation at length two following from eq. (2.41) reads

$$\begin{aligned}\Gamma\left(\begin{smallmatrix}n_1&n_2\\z&0\end{smallmatrix};z\right)&=-(-1)^{n_1}\,\Gamma(n_1+n_2,0;z)+\sum_{r=0}^{n_2}(-1)^{n_1+r}\binom{n_1-1+r}{r}\Gamma(n_2-r,n_1+r;z)\\&\quad+\sum_{r=0}^{n_1}(-1)^{n_1+r}\binom{n_2-1+r}{r}\Gamma(n_1-r,n_2+r;z)+\delta_{n_1,1}\delta_{n_2,1}\zeta_2\ ,\end{aligned} \tag{2.52}$$

and determines $\Gamma\left(\begin{smallmatrix}n_1&n_2\\0&z\end{smallmatrix};z\right)$ through the shuffle identity and eq. (2.44). Analogous relations at length three can be found in appendix B.3.

2.2.3 Relations among elliptic multiple zeta values

Apart from their application to string amplitudes, the above manipulations of eIIs are instrumental to derive relations among eMZVs beyond the obvious reflection and shuffle properties. By definition eq. (2.22), eIIs with all labels $a_j=0$ yield eMZVs in the limit $z\to 1$ of their argument. At the level of labels $a_j=z$, the limit $z\to 1$ is equivalent to $a_j\to 0$ since the $f^{(n)}$ are periodic under $z\mapsto z+1$, hence

$$\lim_{z\to 1}\Gamma\left(\begin{smallmatrix}n_1&n_2&\dots&n_r\\a_1&a_2&\dots&a_r\end{smallmatrix};z\right)=\omega(n_r,\dots,n_2,n_1)\ ,\quad a_j\in\{0,z\}\ ,\quad n_1,n_r\neq 1\ . \tag{2.53}$$

Note that endpoint divergences caused by the simple pole in $f^{(1)}$ might introduce additional MZV constants similar to eq. (2.29), that is why the cases $n_1,n_r=1$ are excluded explicitly.

At length two, for example, eq. (2.52) implies the following eMZV identity provided that the limit $z\to 1$ is non-singular:

$$\begin{aligned}\omega(n_2,n_1)&=-(-1)^{n_1}\omega(0,n_1+n_2)+\sum_{r=0}^{n_2}(-1)^{n_1+r}\binom{n_1-1+r}{r}\omega(n_1+r,n_2-r)\\&\quad+\sum_{r=0}^{n_1}(-1)^{n_1+r}\binom{n_2-1+r}{r}\omega(n_2+r,n_1-r)\ ,\quad n_1,n_2\neq 1\ .\end{aligned} \tag{2.54}$$

At low weights n_i, the coefficients in eq. (2.54) are particularly simple such as

$$\omega(2,3)=\omega(0,5)\ ,\qquad \omega(3,4)=-2\,\omega(0,7)+\omega(2,5)\ . \tag{2.55}$$

Similar procedures can be carried out at higher length. Combining e.g. eq. (B.7) and a suitable generalization thereof to length four leads to

$$0=\omega(0,0,5)+\omega(0,1,4)+\omega(2,0,3) \tag{2.56}$$

$$0=10\,\omega(0,0,0,5)+4\,\omega(0,0,3,2)+2\,\omega(0,2,0,3)-\omega(2)\omega(0,3)-\omega(0,5)\ . \tag{2.57}$$

At length five, a combination of eqs. (2.50) and (2.51) with the shuffle relation eq. (2.23) yields

$$\omega(0,1,0,1,0) = \omega(0,2,0,0,0) \tag{2.58}$$

$$\omega(0,1,1,0,0) = \omega(2,0,0,0,0) - \omega(2)\omega(0,0,0,0)\ , \tag{2.59}$$

which will be applied in subsection 4.3.

3 The functions $f^{(n)}$ on the elliptic curve

In this section, we provide the definition and mathematical framework for the functions $f^{(n)}$, thereby supplementing our heuristic approach in section 2. Before doing so, let us start with some mathematical motivation, in which we explain in particular why we need – in distinction to multiple polylogarithms – an infinite number of them.

3.1 Motivation

The importance of multiple polylogarithms as defined in eq. (2.1) becomes evident, when considering *homotopy-invariant* iterated integrals on the multiply punctured complex plane $\mathbb{C} \setminus \{a_1, ..., a_n\}$: the value of any such integral evaluated on a path γ depends on the homotopy class of the path only and is a $\mathbb{C}$-linear combination of multiple polylogarithms.

Instead of the multiply punctured plane, let us now consider the complex elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with its origin removed (we write this as $E_\tau^\times$), where $\mathrm{Im}\,(\tau) > 0$. One possible definition of multiple elliptic polylogarithms is via iterated integrals on $E_\tau^\times$. Writing the canonical coordinate on $E_\tau^\times$ as $z = s + r\tau$ with $s, r \in \mathbb{R}$, such that $r \equiv \frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}$, two natural differential forms on $E_\tau^\times$ read

$$\mathrm{d}z \quad \text{and} \quad \nu \equiv 2\pi i\, \mathrm{d}r\ . \tag{3.1}$$

These differential forms, however, are not sufficient to describe all iterated integrals on $E_\tau^\times$. Even worse, iterated integrals employing the differential forms $\mathrm{d}z$ and ν only will not be homotopy-invariant in general, i.e. they will depend on the choice of a path in a given homotopy class.

Both problems are overcome simultaneously by supplementing eq. (3.1) by an infinite tower of differentials $f^{(n)}(z)\mathrm{d}z$ constructed through a generating function [16][6]

$$\Omega(z, \alpha, \tau) = \sum_{n\geq 0} f^{(n)}(z)\alpha^{n-1}\ , \tag{3.2}$$

where $f^{(0)}(z) \equiv 1$. In particular, it has been proven in ref. [16] that every iterated integral in ν and $\mathrm{d}z$ can be uniquely lifted to a homotopy-invariant iterated integral over ν and $f^{(n)}(z)\mathrm{d}z$. Conversely, every homotopy-invariant iterated integral on $E_\tau^\times$ arises in this way.

The form of the generating function and its coefficients $f^{(n)}$ in eq. (3.2) can be fixed by constructing a doubly-periodic connection J satisfying the integrability condition

$$\mathrm{d}J + J \wedge J = 0\ . \tag{3.3}$$

This requirement singles out a unique completion of $J = \nu X_0 + \mathrm{d}z\, X_1 + \dots$ to a formal power series in non-commuting variables X_0 and X_1 given by [16]

$$J = \nu X_0 - \mathrm{ad}_{X_0}\Omega(z, -\mathrm{ad}_{X_0}, \tau)(X_1)\mathrm{d}z\ . \tag{3.4}$$

[6] Note that in ref. [16], $\Omega(z, \alpha, \tau)$ is defined as a differential form, i.e. includes $\mathrm{d}z$.

It follows from eq. (3.3) that every word in X_0, X_1 in the formal power series

$$\sum_{k=0}^{\infty} \int J^k \tag{3.5}$$

is a homotopy-invariant iterated integral on $E_\tau^\times$, and one can prove that in fact every such iterated integral arises in this way. Therefore, every homotopy invariant iterated integral on $E_\tau^\times$ can be written as a special linear combination of iterated integrals of the differential forms $f^{(n)}(z)\mathrm{d}z$ and ν. The differential form ν eq. (3.1), however, vanishes on the real integration path $\gamma(t) \in \mathbb{R}$. Hence, the setup in subsection 2.2 based on real variables leads to elliptic multiple zeta values defined in ref. [17] without referring to the differential form ν.

Although homotopy invariance is generically lost for the iterated integral over the forms $f^{(n_1)}(z_1)\mathrm{d}z_1 \ldots f^{(n_r)}(z_r)\mathrm{d}z_r$ on the punctured elliptic curve $E_\tau^\times$, its value at the real path $[0,1]$ as in eq. (2.22) can in fact be written as a $\mathbb{Z}$-linear combination of coefficients of words in eq. (3.5), again evaluated on the path $[0,1]$. In particular, this shows that the eMZVs associated with the path $[0,1]$ [17] are periods of the fundamental group of $E_\tau^\times$.

Hence, the eIIs defined by eq. (2.18) coincide with the elliptic polylogarithms defined in ref. [16] when restricted to the real line. They can be lifted to honest homotopy-invariant iterated integrals on the punctured elliptic curve by means of the differential form ν defined in eq. (3.1). However, generic combinations of $f^{(n)}(z)\mathrm{d}z$ accompany several words in X_0, X_1 in eq. (3.5) and therefore allow for various homotopy-invariant completions using ν. Iterated integrals over ν and $\mathrm{d}z$, on the other hand, correspond to a single word in eq. (3.5) and therefore have a unique uplift via $f^{(n\geq 1)}(z)\mathrm{d}z$ towards the elliptic polylogarithms of ref. [16].

3.2 Doubly-periodic functions and generating series

In this section, we define the functions $f^{(n)}$ through a generating series, closely following ref. [16]. In the sequel, z and α are complex coordinates on $E_\tau^\times$. Simultaneously, α will be used as a formal expansion variable below. The modular parameter often appears in the combination

$$q \equiv e^{2\pi i \tau} \ , \tag{3.6}$$

where $\mathrm{Im}\,(\tau) > 0$ translates into $|q| < 1$, relevant for convergence issues.

3.2.1 Some doubly-periodic functions

A general reference on doubly-periodic functions is ref. [23]. Let θ_1 denote the odd Jacobi function[7] defined by

$$\theta_1(z,\tau) \equiv 2iq^{1/8} \sin(\pi z) \prod_{j=1}^{\infty}(1-q^j) \prod_{j=1}^{\infty}(1-e^{2\pi i z}q^j) \prod_{j=1}^{\infty}(1-e^{-2\pi i z}q^j) \ , \tag{3.7}$$

subject to the following periodicity properties

$$\theta_1(z+1,\tau) = -\theta_1(z,\tau) \ , \qquad \theta_1(z+\tau,\tau) = -e^{-\pi i \tau} e^{-2\pi i z} \theta_1(z,\tau) \ . \tag{3.8}$$

[7] The subsequent definitions of $f^{(n)}$ are unchanged by z-independent rescalings of θ_1. Hence, the current setup is consistent with refs. [16, 50], which rely on $\theta(z,\tau) \equiv 2iq^{1/12} \sin(\pi z) \prod_{j=1}^{\infty}(1-e^{2\pi i z}q^j) \prod_{j=1}^{\infty}(1-e^{-2\pi i z}q^j)$.

For $j \geq 1$ we also define the *Eisenstein function* $E_j(z,\tau)$ and the *Eisenstein series* $e_j(\tau)$ by[8]

$$E_j(z,\tau) \equiv \sum_{m,n\in\mathbb{Z}} \frac{1}{(z+m+n\tau)^j} \qquad e_j(\tau) \equiv \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^j} \tag{3.9}$$

which are related to the function $\theta_1(z,\tau)$ via

$$\frac{\partial}{\partial z}\ln(\theta_1(z,\tau)) = E_1(z,\tau)\ , \qquad \frac{\partial}{\partial z}E_j(z,\tau) = -jE_{j+1}(z,\tau)\ . \tag{3.10}$$

3.2.2 The Eisenstein-Kronecker series

The Eisenstein-Kronecker series $F(z,\alpha,\tau)$ is defined by [51, 16]

$$F(z,\alpha,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}\ , \tag{3.11}$$

where $'$ denotes a derivative with respect to the first argument. Taking the logarithmic derivative of eq. (3.11) together with the Taylor expansion $E_1(\alpha,\tau) = \frac{1}{\alpha} - \sum_{j=0}^{\infty}\alpha^j e_{j+1}(\tau)$ leads to the following alternative representation [52, 50]

$$F(z,\alpha,\tau) = \frac{1}{\alpha}\exp\left(-\sum_{j\geq 1}\frac{(-\alpha)^j}{j}(E_j(z,\tau)-e_j(\tau))\right) \tag{3.12}$$

in terms of the Eisenstein functions and Eisenstein series defined in eq. (3.9). The periodicity properties of the θ_1-function in eq. (3.8) imply that the Eisenstein-Kronecker series is quasi-periodic,

$$F(z+1,\alpha,\tau) = F(z,\alpha,\tau), \qquad F(z+\tau,\alpha,\tau) = e^{-2\pi i\alpha}F(z,\alpha,\tau)\ . \tag{3.13}$$

Moreover, the representation (3.12) together with the Fay trisecant equation [53] yields the Fay identity

$$\begin{aligned} F(z_1,\alpha_1,\tau)F(z_2,\alpha_2,\tau) &= F(z_1,\alpha_1+\alpha_2,\tau)F(z_2-z_1,\alpha_2,\tau) \\ &\quad + F(z_2,\alpha_1+\alpha_2,\tau)F(z_1-z_2,\alpha_1,\tau)\ . \end{aligned} \tag{3.14}$$

3.2.3 Restoring double periodicity and modularity

The quasi-periodicity of the Eisenstein-Kronecker series under $z \to z+\tau$ as given in eq. (3.13) can be lifted to an honest periodic behavior by defining

$$\Omega(z,\alpha,\tau) \equiv \exp\left(2\pi i\alpha\frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}\right)F(z,\alpha,\tau)\,. \tag{3.15}$$

Clearly, the resulting function $\Omega(z,\alpha,\tau)$ is doubly-periodic in z,

$$\Omega(z+1,\alpha,\tau) = \Omega(z+\tau,\alpha,\tau) = \Omega(z,\alpha,\tau)\ , \tag{3.16}$$

[8] The two cases $j = 1, 2$ require the Eisenstein summation prescription

$$\sum_{m,n\in\mathbb{Z}} a_{m,n} \equiv \lim_{N\to\infty}\lim_{M\to\infty}\sum_{n=-N}^{N}\sum_{m=-M}^{M} a_{m,n}\ .$$

and holomorphicity of the Eisenstein-Kronecker series eq. (3.11) gives rise to the differential equation

$$\frac{\partial}{\partial \bar{z}}\Omega(z,\alpha,\tau) = -\frac{\pi\alpha}{\mathrm{Im}\,(\tau)}\Omega(z,\alpha,\tau)\,. \tag{3.17}$$

The latter implies that the connection J in eq. (3.4) satisfies the integrability condition eq. (3.3) and generates homotopy-invariant iterated integrals via the formal power series eq. (3.5) [16].

Upon taking the exponential in eq. (3.15) into account, the modular transformation properties of the Eisenstein-Kronecker series [52, 54], can be translated into

$$\Omega\Big(\frac{z}{c\tau+d}, \frac{\alpha}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)\Omega(z,\alpha,\tau) \tag{3.18}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. The Fay identity eq. (3.14) for the Eisenstein-Kronecker series carries over to

$$\begin{aligned}\Omega(z_1,\alpha_1,\tau)\Omega(z_2,\alpha_2,\tau) &= \Omega(z_1,\alpha_1+\alpha_2,\tau)\Omega(z_2-z_1,\alpha_2,\tau)\\ &\quad + \Omega(z_2,\alpha_1+\alpha_2,\tau)\Omega(z_1-z_2,\alpha_1,\tau)\end{aligned} \tag{3.19}$$

after multiplication with $\exp\big(\frac{2\pi i}{\mathrm{Im}\,(\tau)}\big[\alpha_1 \mathrm{Im}\,(z_1) + \alpha_2 \mathrm{Im}\,(z_2)\big]\big)$.

3.3 Definition and properties of the weighting functions $f^{(n)}$

3.3.1 Definition of $f^{(n)}$

We define the functions $f^{(n)}$ entering the eIIs eq. (2.18) through the following Taylor series in α,

$$\alpha\Omega(z,\alpha,\tau) \equiv \sum_{n=0}^{\infty} f^{(n)}(z,\tau)\alpha^n\ . \tag{3.20}$$

They are real analytic on the punctured elliptic curve $E_\tau^\times$. As above, we will omit the argument τ and write $f^{(n)}(z)$ or often simply $f^{(n)}$. Their explicit form is conveniently captured by the following functions[9] $\mathcal{E}_n$

$$\mathcal{E}_1(z,\tau) \equiv E_1(z,\tau) + 2\pi i\frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}\ , \qquad \mathcal{E}_n(z,\tau) \equiv (-1)^n\big(e_n(\tau) - E_n(z,\tau)\big)\ \forall\ n\geq 2\ . \tag{3.21}$$

These functions result in a simple representation of the generating series

$$\alpha\Omega(z,\alpha,\tau) = \exp\left(\sum_{j=1}^{\infty}\frac{\alpha^j}{j}\mathcal{E}_j(z,\tau)\right)\ , \tag{3.22}$$

and allow for a combinatorial interpretation of $f^{(n)}(z,\tau)$ in terms of the cycle index of the symmetric group S_n (see appendix D).

Comparison with eq. (3.20) yields the following expressions for the lowest functions $f^{(n)}$

$$\begin{aligned} f^{(1)} &= \mathcal{E}_1\\ f^{(2)} &= \frac{1}{2}(\mathcal{E}_1^2 + \mathcal{E}_2)\end{aligned}$$

[9]Note that all $\mathcal{E}_n$ are meromorphic except for $\mathcal{E}_1$ (due to the term $\mathrm{Im}\,(z)$), and that $\mathcal{E}_2(z) = -\wp(z)$ is the Weierstrass function. Higher functions $\mathcal{E}_n$ at $n\geq 3$ are related to derivatives of the Weierstrass function, e.g. $\mathcal{E}_3 = -\frac{1}{2}\partial\wp$ and $\mathcal{E}_4 = e_4 - \frac{1}{6}\partial^2\wp$.

$$
\begin{aligned}
f^{(3)} &= \frac{1}{3!}(\mathcal{E}_1^3 + 3\mathcal{E}_1\mathcal{E}_2 + 2\mathcal{E}_3) \\
f^{(4)} &= \frac{1}{4!}(\mathcal{E}_1^4 + 6\mathcal{E}_1^2\mathcal{E}_2 + 8\mathcal{E}_1\mathcal{E}_3 + 3\mathcal{E}_2^2 + 6\mathcal{E}_4) \\
f^{(5)} &= \frac{1}{5!}(\mathcal{E}_1^5 + 10\mathcal{E}_1^3\mathcal{E}_2 + 20\mathcal{E}_1^2\mathcal{E}_3 + 15\mathcal{E}_1\mathcal{E}_2^2 + 30\mathcal{E}_1\mathcal{E}_4 + 20\mathcal{E}_2\mathcal{E}_3 + 24\mathcal{E}_5) \ .
\end{aligned} \tag{3.23}
$$

The functions $\mathcal{E}_j$ can be expressed in terms of $\ln\theta_1$ via eq. (3.10), which leads to the representations for $f^{(1)}$ and $f^{(2)}$ provided in eqs. (2.15) and (2.16). As shown in appendix D, the general expression for $f^{(n)}$ following from eq. (3.22) reads

$$
f^{(n)} = \sum_{a_1,a_2,\ldots,a_n\geq 0} \delta\left(\sum_{i=1}^{n} i a_i - n\right) \prod_{j=1}^{n} \frac{\mathcal{E}_j^{a_j}}{j^{a_j} a_j!} \ , \tag{3.24}
$$

and an equivalent recursive representation is given by

$$
f^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \mathcal{E}_j f^{(n-j)} \ . \tag{3.25}
$$

3.3.2 Properties of $f^{(n)}$

The functions $f^{(n)}$ inherit their double periodicity, the form of their antiholomorphic derivative as well as their behavior under modular transformations from the generating series in eqns. (3.16), (3.17) and (3.18):

$$
f^{(n)}(z+1) = f^{(n)}(z+\tau) = f^{(n)}(z) \tag{3.26}
$$

$$
\frac{\partial f^{(n)}(z)}{\partial \bar{z}} = -\frac{\pi}{\mathrm{Im}\,(\tau)} f^{(n-1)}(z) \tag{3.27}
$$

$$
f^{(n)}\Big(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^n f^{(n)}(z,\tau) \ . \tag{3.28}
$$

Likewise, the Fay identity eq. (3.19) implies for $f^{(n)}_{ij} \equiv f^{(n)}(z_i - z_j)$:

$$
f_{il}^{(m-1)} f_{jl}^{(n)} + f_{il}^{(m)} f_{jl}^{(n-1)} = \sum_{r=0}^{n} \binom{m-1+r}{r} f_{ji}^{(n-r)} f_{il}^{(m-1+r)} + \sum_{r=0}^{m} \binom{n-1+r}{r} f_{ij}^{(m-r)} f_{jl}^{(n-1+r)} \ . \tag{3.29}
$$

This identity has been used repeatedly to derive relations among eIIs in section 2 (cf. eq. (2.40) above).

Given the singular factor $\frac{\theta_1'(0,\tau)}{\theta_1(z,\tau)} = \frac{1}{z} + \mathcal{O}(z)$ in the Eisenstein-Kronecker series eq. (3.11), one can check that the residue at the simple pole of Ω at the origin is independent on α. Hence, only $f^{(1)}$ has a simple pole at any $z = k + \tau l$ for $k, l \in \mathbb{Z}$ whereas all other weighting functions $f^{(n\neq 1)}$ are regular on the entire elliptic curve:

$$
\lim_{z\to 0} z f^{(n)}(z) = \delta_{n,1} \ . \tag{3.30}
$$

It is this property of the functions $f^{(n)}$, which is responsible for the $z \to 0$ behavior stated in eq. (2.29).

3.3.3 q-expansions of $f^{(n)}$

The Eisenstein-Kronecker series eq. (3.11) is known to have the following power-series expansion in $q = e^{2\pi i\tau}$ [23, 16]

$$\alpha F(z,\alpha,\tau) = 1 + \pi\alpha\cot(\pi z) - 2\sum_{k=1}^{\infty}\zeta_{2k}\alpha^{2k} - 2\pi i\alpha\sum_{m,n=1}^{\infty}\Big(e^{2\pi i(mz+n\alpha)} - e^{-2\pi i(mz+n\alpha)}\Big)q^{mn}$$
$$\equiv \sum_{n=0}^{\infty} g^{(n)}(z)\alpha^n \ . \tag{3.31}$$

Disentangling the powers of α yields the holomorphic parts $g^{(n)}$ of the functions $f^{(n)}$, e.g.

$$g^{(1)}(z) = \pi\cot(\pi z) + 4\pi\sum_{m=1}^{\infty}\sin(2\pi mz)\sum_{n=1}^{\infty}q^{mn} \tag{3.32}$$
$$g^{(2)}(z) = -2\zeta_2 + 8\pi^2\sum_{m=1}^{\infty}\cos(2\pi mz)\sum_{n=1}^{\infty}nq^{mn} \tag{3.33}$$
$$g^{(3)}(z) = -8\pi^3\sum_{m=1}^{\infty}\sin(2\pi mz)\sum_{n=1}^{\infty}n^2q^{mn} \ , \tag{3.34}$$

where $\cot(\pi z) = \frac{1}{\pi z} + \mathcal{O}(z)$ captures the simple pole of $f^{(1)}$. More generally, we find

$$g^{(k)}(z)\Big|_{k=2,4,\dots} = -2\Big[\zeta_k + \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\cos(2\pi mz)\sum_{n=1}^{\infty}n^{k-1}q^{mn}\Big] \tag{3.35}$$
$$g^{(k)}(z)\Big|_{k=3,5,\dots} = -2i\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sin(2\pi mz)\sum_{n=1}^{\infty}n^{k-1}q^{mn} \ . \tag{3.36}$$

The non-holomorphic piece in $f^{(n)}$ consisting of factors $\frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}$ can be immediately restored via

$$f^{(n)}(z) = \sum_{k=0}^{n}\frac{\big[2\pi i\mathrm{Im}\,(z)\big]^k}{k!\big[\mathrm{Im}\,(\tau)\big]^k}g^{(n-k)}(z) \ . \tag{3.37}$$

Even though the functions $f^{(n)}$ in the definition eq. (2.18) of eIIs are evaluated at real arguments in the subsequent, we will keep track of the admixtures of $\mathrm{Im}\,(z)$ in eq. (3.37) for further applications beyond this work. For example, another system of eIIs and eMZVs can be defined for the path from 0 to τ instead of the real interval $[0,1]$ whose properties are crucially affected by the factors of $\mathrm{Im}\,(z)$ and the resulting modular properties.

4 The one-loop four-point amplitude in open string theory

Iterated integrals defined on an elliptic curve in subsection 2.2 appear naturally in superstring theory. Calculating one-loop scattering amplitudes among open string states amounts to evaluating iterated integrals weighted by the functions $f^{(n)}$ defined in section 3. Accordingly, the expansion of one-loop superstring amplitudes in the inverse string tension α' involves eMZVs.

The α'-expansion of tree-level amplitudes in open string theory is well known to involve standard MZVs, see e.g. ref. [34]. The pattern of their appearance is much simpler as compared to the MZVs and polylogarithms in loop amplitudes of field theory and can be understood in terms of motivic MZVs [29] as well as the Drinfeld associator [38]. Hence, it is not surprising

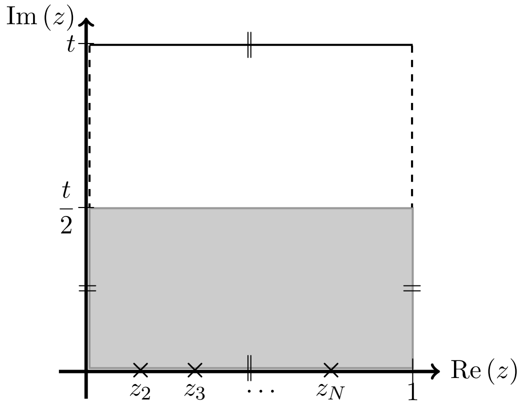

Figure 1: Parametrization of the cylinder worldsheet through the shaded region. The boundary under investigation has real coordinates $z_j \in [0, 1]$. The identified edges inherited from the underlying torus at $\tau = it$ are marked by $=$ and $\|$, respectively.

that one-loop string amplitudes furnish a perfect laboratory to study patterns and properties of eMZVs.

Iterated integrals in one-loop open string amplitudes occur on the boundaries of a two-dimensional worldsheet of either cylinder or Möbius-strip topology [40]. They describe conformally inequivalent configurations of inserting open string states on the respective boundaries. As a first field of application for eMZVs, we will entirely focus on cylindrical worldsheets in this work with all integrations confined to one boundary[10]. As shown in figure 1, this situation can be described by a torus with purely imaginary modular parameter $\tau = it$ with $t \in \mathbb{R}$. The cylinder boundaries are then parametrized by $\mathrm{Re}\,(z_j) \in [0, 1]$ with $\mathrm{Im}\,(z_j) = 0$ and $\mathrm{Im}\,(z_j) = \frac{t}{2}$, respectively. The configuration of interest with one boundary empty is captured by real insertion points $z_j \in \mathbb{R}$.

4.1 The four-point amplitude

For massless open-string excitations in ten dimensions – gluons and gluinos – supersymmetry requires at least four external states for a non-vanishing one-loop amplitude, so the simplest case to be studied below is the four-point function [58, 59],

$$A^{\text{1-loop}}_{\text{string}}(1, 2, 3, 4) = s_{12} s_{23} A^{\text{tree}}_{\text{YM}}(1, 2, 3, 4) \int_0^\infty \mathrm{d}t \; I_{4\text{pt}}(1, 2, 3, 4) \tag{4.1}$$

$$I_{4\text{pt}}(1, 2, 3, 4) \equiv \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \int_0^{z_2} \mathrm{d}z_1 \, \delta(z_1) \prod_{j<k}^{4} \exp\left[s_{jk} P_{jk}\right] . \tag{4.2}$$

The entire polarization dependence is captured by the four-point tree amplitude of sYM field theory, see [60] for its tensor structure. The worldsheet integral $I_{4\text{pt}}(1, 2, 3, 4)$ depends on the external momenta k_i through dimensionless Mandelstam invariants

$$s_{ij} \equiv \alpha'(k_i + k_j)^2 \; , \tag{4.3}$$

[10] The interplay between open string worldsheets of different topologies is crucial for the cancellations of infinities [55] and anomalies [56, 57] which occur for gauge group $SO(32)$.

where momentum conservation and the mass-shell condition $k_i^2 = 0$ leave two independent s_{ij},

$$s_{34} = s_{12} \ , \qquad s_{14} = s_{23} \ , \qquad s_{13} = s_{24} = -s_{12} - s_{23} \ . \tag{4.4}$$

The dependence on worldsheet positions $z_j \in [0,1]$ enters through the genus-one Green function

$$P_{ij} \equiv \ln \left| \frac{\theta_1(z_i - z_j, \tau)}{\theta_1'(0,\tau)} \right|^2 - \frac{2\pi}{\mathrm{Im}\,(\tau)} \big[\mathrm{Im}\,(z_i - z_j)\big]^2 \tag{4.5}$$

which is related to the singular function $f_{ij}^{(1)} \equiv f^{(1)}(z_i - z_j)$ in eq. (2.15) via

$$\partial P_{ij} = f_{ij}^{(1)} \ , \qquad P_{ij} = \int_{z_j}^{z_i} \mathrm{d}w \, f^{(1)}(w - z_j) \ . \tag{4.6}$$

The endpoint divergence as $w \to z_j$ can be dealt with through the regularization prescription eq. (2.26) which heuristically amounts to $\lim_{z_i \to z_j} P_{ij} = 0$. Note that the dependence of $I_{4\mathrm{pt}}(1,2,3,4)$ on s_{ij} and $q \equiv e^{-2\pi t}$ is suppressed for ease of notation.

The non-holomorphic piece in $f^{(1)}(z) \equiv \partial \ln \theta_1(z,\tau) + 2\pi i \frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}$ drops out for the present cylinder parametrization where all vertices are inserted on the boundary with real coordinates z_j. Accordingly, the differential form $\nu \sim \mathrm{d}\,\mathrm{Im}\,(z)$ in eq. (3.1) required for homotopy invariance does not contribute to the cylinder integrals under consideration. However, the admixtures of $\frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}$ in $f^{(n)}$ are crucial for modular invariance of closed-string amplitudes and cylinder diagrams with open string states on both boundaries.

Translation invariance on genus-one surfaces can be used to fix $z_1 = 0$. In addition, the N-point integration measure which appears for $N = 4$ in eq. (4.2),

$$\int_{12\ldots N} \equiv \int_0^1 \mathrm{d}z_N \int_0^{z_N} \mathrm{d}z_{N-1} \ldots \int_0^{z_3} \mathrm{d}z_2 \int_0^{z_2} \mathrm{d}z_1 \, \delta(z_1) \ , \tag{4.7}$$

is invariant under cyclic shifts $z_i \to z_{i+1 \bmod N}$ and, up to a sign $(-1)^N$, under reflection $z_i \to z_{N+1-i}$. Some features of the one-loop N-point amplitudes are discussed in section 5. Their integrand then involves factors of $f^{(w_i)}(z_j - z_k)$ with overall weight $\sum_i w_i = N - 4$.

As another generalization of the one-loop amplitude eq. (4.1) in ten spacetime dimensions, one could consider supersymmetry-preserving compactifications on a torus. For each circular dimension of radius R, the associated momentum components are quantized and contribute a correction factor of $\sum_{n=-\infty}^{\infty} e^{-n^2 \pi t R^2/\alpha'}$ to the t-integrand [61]. Since this does not affect the z_j-integrations within $I_{4\mathrm{pt}}(1,2,3,4)$ and the resulting eMZVs, the subsequent results on the α'-expansion are universal for any torus compactification to spacetime dimensions $D \leq 10$.

4.2 The α'-expansion

In this section, we investigate the α'-expansion of the t-integrand in eq. (4.1),

$$I_{4\mathrm{pt}}(1,2,3,4) = \int_{1234} \prod_{i<j}^{4} \sum_{n_{ij}=0}^{\infty} \frac{1}{n_{ij}!} (s_{ij} P_{ij})^{n_{ij}} \ , \tag{4.8}$$

which encodes the low-energy effective action for the gluon supermultiplet. Expanding in α' amounts to Taylor expanding the exponential in eq. (4.2) in all the Mandelstam invariants s_{ij} defined in eq. (4.3) as well as the corresponding worldsheet Green function P_{ij} given by eq. (4.6).

In addition to the power-series expansion in α' discussed in the subsequent, the integration region of large t in the amplitude eq. (4.1) gives rise to logarithmic, non-analytic momentum dependence. The associated threshold singularities in s_{ij} are for instance crucial to make contact with the Feynman box integral in the sYM amplitude arising in the point-particle limit [61]. Mimicking the low energy-analysis of closed string one-loop amplitudes [62–65], we separate the analytic from the non-analytic parts of the amplitude and do not keep track of the non-analytic threshold singularities.

The simplest monomials in P_{ij} inequivalent under cyclic shifts and reflections of the vertex positions z_j integrate to

$$c_0 \equiv \int_{1234} 1 \ , \qquad c_1^1 \equiv \int_{1234} P_{12} \ , \qquad c_2^1 \equiv \int_{1234} P_{13} \ . \tag{4.9}$$

At second and third order in α' one finds

$$\begin{aligned} c_1^2 &\equiv \frac{1}{2}\int_{1234} P_{12}^2 \ , \qquad c_3^2 \equiv \int_{1234} P_{12}P_{14} \ , \qquad c_5^2 \equiv \int_{1234} P_{12}P_{34} \\ c_2^2 &\equiv \frac{1}{2}\int_{1234} P_{13}^2 \ , \qquad c_4^2 \equiv \int_{1234} P_{13}P_{24} \ , \qquad c_6^2 \equiv \int_{1234} P_{12}P_{13} \end{aligned} \tag{4.10}$$

as well as

$$\begin{aligned} c_1^3 &\equiv \frac{1}{6}\int_{1234} P_{12}^3 \ , & c_5^3 &\equiv \frac{1}{2}\int_{1234} P_{12}^2 P_{34} \ , & c_9^3 &\equiv \int_{1234} P_{12}P_{13}P_{23} \\ c_2^3 &\equiv \frac{1}{6}\int_{1234} P_{13}^3 \ , & c_6^3 &\equiv \frac{1}{2}\int_{1234} P_{12}^2 P_{13} \ , & c_{10}^3 &\equiv \int_{1234} P_{12}P_{13}P_{14} \\ c_3^3 &\equiv \frac{1}{2}\int_{1234} P_{12}^2 P_{23} \ , & c_7^3 &\equiv \frac{1}{2}\int_{1234} P_{12}P_{13}^2 \ , & c_{11}^3 &\equiv \int_{1234} P_{12}P_{34}P_{13} \\ c_4^3 &\equiv \frac{1}{2}\int_{1234} P_{13}^2 P_{24} \ , & c_8^3 &\equiv \int_{1234} P_{12}P_{23}P_{34} \ , & c_{12}^3 &\equiv \int_{1234} P_{13}P_{24}P_{12} \ . \end{aligned} \tag{4.11}$$

As will be demonstrated in section 4.3, eMZVs defined in eq. (2.22) are the natural language to describe the above c_i^j and to understand the linear combinations appearing after applying momentum conservation eq. (4.4):

$$\begin{aligned} I_{\rm 4pt}(1,2,3,4) &= c_0 \ + \ 2(c_1^1-c_2^1)\,(s_{12}+s_{23}) \ + \ (2c_1^2+2c_2^2-c_3^2-c_4^2)\,(s_{12}^2+\tfrac{1}{4}s_{12}s_{23}+s_{23}^2) \\ &+ \ \frac{1}{4}\,(-2c_1^2+14c_2^2+c_3^2-7c_4^2)\,s_{12}s_{23} \ + \ 2\,(c_{10}^3-2c_1^3-c_2^3+2c_3^3+c_4^3-2c_9^3)\,s_{12}s_{23}(s_{12}+s_{23}) \\ &+ \ (2c_{10}^3+2c_1^3-2c_2^3+6c_3^3+2c_4^3-8c_6^3-2c_8^3)\,(s_{12}+s_{23})(s_{12}^2+s_{12}s_{23}+s_{23}^2) \ + \ \mathcal{O}(\alpha'^4)\,. \end{aligned} \tag{4.12}$$

A first flavor of relations among c_i^j (and thus ultimately among eMZVs) can be obtained by exploiting cyclic and reflection properties of five-point integrals such as

$$\int_{12345} P_{45}\partial_2 P_{23} = \int_{12345} P_{51}\partial_2 P_{23} \quad \Rightarrow \quad \int_{1345} P_{45}P_{13} = \int_{1345} P_{51}P_{13} \quad \Rightarrow \quad c_3^2 = c_5^2 \ , \tag{4.13}$$

see eq. (4.7) for the measure $\int_{12345}$. Similar methods imply that

$$2c_6^2 = c_3^2 + c_4^2 \ , \quad c_3^3 = c_5^3 \ , \quad c_{10}^3 = c_{11}^3 \ , \quad c_7^3 + c_6^3 = c_3^3 + c_4^3 \ , \quad c_{11}^3 + c_{10}^3 = c_8^3 + c_{12}^3 \ , \tag{4.14}$$

these relations have been used to eliminate c_5^2, c_6^2 as well as $c_5^3, c_7^3, c_{12}^3, c_{11}^3$ from eq. (4.12).

Note that the α'-expansion of closed string one-loop amplitudes has been analyzed along similar lines in refs. [62–65]. Since each closed-string insertion point z_j is integrated over the entire torus E_τ, integrals involving propagators with a free endpoint vanish and therefore much fewer closed-string counterparts of the coefficients c_i^j arise.

4.3 Elliptic multiple zeta values

In this section we convert the constituents of the α'-expansion, c_i^j defined by eqns. (4.9), (4.10) and (4.11), to eMZVs. This will provide a characterization of the particular linear combinations of c_i^j which appear in eq. (4.12) along with various powers of s_{12} and s_{23}.

The leading term c_0 in eq. (4.9) can be straightforwardly evaluated to yield $\frac{1}{6}$ and furnishes a special case of

$$\omega(\underbrace{0,0,\ldots,0}_{n}) = \frac{1}{n!}\,, \tag{4.15}$$

which follows from multiple insertions of $1 = f^{(0)}(z_i)$. Nevertheless, it will prove instructive for the comparison with higher orders in α' to express c_0 as an unevaluated eMZV:

$$\begin{aligned} c_0 &= \int_0^1 f^{(0)}(z_4)\,\mathrm{d}z_4 \int_0^{z_4} f^{(0)}(z_3)\,\mathrm{d}z_3 \int_0^{z_3} f^{(0)}(z_2)\,\mathrm{d}z_2 \\ &= \Gamma(0,0,0;1) = \omega(0,0,0)\,. \end{aligned} \tag{4.16}$$

Below, we will repeatedly apply the definitions eq. (2.18) and eq. (2.22) of eIIs and eMZVs, respectively, in order to express the other integrals c_i^j in the same fashion.

4.3.1 First order in P_{ij}: integrals c_i^1

At linear order in s_{ij}, we substitute $P_{1j} = \int_0^{z_j} f^{(1)}(w)\,\mathrm{d}w$ according to eq. (4.6) and $z_1 = 0$ into the definitions eq. (4.9) and find

$$\begin{aligned} c_1^1 &= \int_0^1 f^{(0)}(z_4)\,\mathrm{d}z_4 \int_0^{z_4} f^{(0)}(z_3)\,\mathrm{d}z_3 \int_0^{z_3} f^{(0)}(z_2)\,\mathrm{d}z_2 \int_0^{z_2} f^{(1)}(w)\,\mathrm{d}w \\ &= \Gamma(0,0,0,1;1) = \omega(1,0,0,0) \end{aligned} \tag{4.17}$$

$$\begin{aligned} c_2^1 &= \int_0^1 f^{(0)}(z_4)\,\mathrm{d}z_4 \int_0^{z_4} f^{(0)}(z_3)\,\mathrm{d}z_3 \int_0^{z_3} f^{(0)}(z_2)\,\mathrm{d}z_2 \int_0^{z_3} f^{(1)}(w)\,\mathrm{d}w \\ &= \int_0^1 f^{(0)}(z_4)\,\mathrm{d}z_4 \int_0^{z_4} f^{(0)}(z_3)\,\mathrm{d}z_3\ \Gamma(0;z_3)\ \Gamma(1;z_3) \\ &= \int_0^1 f^{(0)}(z_4)\,\mathrm{d}z_4 \int_0^{z_4} f^{(0)}(z_3)\,\mathrm{d}z_3\,\big[\Gamma(1,0;z_3) + \Gamma(0,1;z_3)\big] \\ &= \Gamma(0,0,0,1;1) + \Gamma(0,0,1,0;1) = \omega(1,0,0,0) + \omega(0,1,0,0)\ . \end{aligned} \tag{4.18}$$

The second line of eq. (4.18) makes use of the shuffle product eq. (2.19) for eIIs. Equivalence of eq. (4.17) with the cyclically shifted integrand

$$\begin{aligned} \int_{1234} P_{14} &= \int_0^1 f^{(0)}(z_4)\,\mathrm{d}z_4 \int_0^{z_4} f^{(0)}(z_3)\,\mathrm{d}z_3 \int_0^{z_3} f^{(0)}(z_2)\,\mathrm{d}z_2 \int_0^{z_4} f^{(1)}(w)\,\mathrm{d}w \\ &= \omega(1,0,0,0) + \omega(0,1,0,0) + \omega(0,0,1,0) \end{aligned} \tag{4.19}$$

can be checked using antisymmetry $\omega(0,1,0,0) + \omega(0,0,1,0) = 0$ following from eq. (2.24).

4.3.2 Second order in P_{ij}: integrals c_i^2

At quadratic order in s_{ij}, the rewriting $P_{1j} = \int_0^{z_j} f^{(1)}(w)\,\mathrm{d}w = -\int_{z_j}^1 f^{(1)}(w)\,\mathrm{d}w$ allows to straightforwardly address any quadratic monomial in P_{12}, P_{13}, P_{14} along the lines of eqs. (4.17) and (4.18):

$$c_1^2 = \omega(1,1,0,0,0) \tag{4.20a}$$
$$c_2^2 = \omega(1,1,0,0,0) + \omega(1,0,1,0,0) + \omega(0,1,1,0,0) \tag{4.20b}$$
$$c_3^2 = -\omega(1,0,0,0,1) \tag{4.20c}$$
$$c_6^2 = 2\omega(1,1,0,0,0) + \omega(1,0,1,0,0)\ . \tag{4.20d}$$

Then, eqs. (4.13) and (4.14) can be used to determine the remaining two c_j^2 in eq. (4.10):

$$c_4^2 = 2\omega(1,1,0,0,0) + \omega(1,0,1,0,0) - \omega(1,0,0,1,0) \tag{4.21a}$$
$$c_5^2 = -\omega(1,0,0,0,1)\ . \tag{4.21b}$$

Note that the integration limits $\int_0^{z_j}\ldots$ in the representation of P_{1j} can be traded for $-\int_{z_j}^1\ldots$. This is equivalent to applying a shuffle relation eq. (2.23),

$$0 = \omega(1)\omega(1,0,0,0) = 2\omega(1,1,0,0,0) + \omega(1,0,1,0,0) + \omega(1,0,0,1,0) + \omega(1,0,0,0,1) \tag{4.22}$$
$$0 = \omega(1)\omega(0,1,0,0) = \omega(1,0,1,0,0) + 2\omega(0,1,1,0,0) + \omega(0,1,0,1,0) + \omega(0,1,0,0,1)\ , \tag{4.23}$$

where $\omega(1)$ vanishes by the reflection identity eq. (2.24).

4.3.3 Integration techniques for P_{23}, P_{24}, P_{34}

Green functions P_{ij} where both indices describe a leg to be integrated (legs $2,3,4$) are more difficult to integrate. Their integral representation eq. (4.6) inevitably gives rise to iterated integrals $\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_r \\ a_1 & \dots & a_r \end{smallmatrix}; z\right)$ with the argument appearing in the labels, that is $a_i = z$. Integration over z_3 and z_4 then requires the techniques of subsection 2.2.2, in particular the recursion formulæ eq. (2.41) to eq. (2.43).

The simple corollary $\Gamma\left(\begin{smallmatrix} 1 & 0 \\ z & 0 \end{smallmatrix}; z\right) = -\Gamma\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}; z\right)$ of the reflection identity eq. (2.20) is sufficient to integrate P_{23} and to reproduce eq. (4.17) from a different cyclic representative. The quadratic case $c_5^2 = \int_{1234} P_{12}P_{34}$, on the other hand, requires more effort. One obtains

$$\begin{aligned}\int_{1234} P_{12}P_{34} &= -\int_0^1 f^{(0)}\mathrm{d}z_4 \int_0^{z_4} f^{(1)}(w-z_4)\mathrm{d}w \int_0^{w} f^{(0)}\mathrm{d}z_3 \int_0^{z_3} f^{(0)}\mathrm{d}z_2 \int_0^{z_2} f^{(1)}(u)\mathrm{d}u \\ &= -\int_0^1 f^{(0)}(z_4)\mathrm{d}z_4\ \Gamma\left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ z_4 & 0 & 0 & 0 \end{smallmatrix}; z_4\right) \\ &= 2\omega(1,1,0,0,0) - 2\omega(2,0,0,0,0) - \omega(0,2,0,0,0) - \zeta_2\omega(0,0,0)\ ,\end{aligned} \tag{4.24}$$

where $\Gamma\left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ z_4 & 0 & 0 & 0 \end{smallmatrix}; z_4\right)$ has been reexpressed via eq. (2.49) in the last step. In order to reproduce the result of eq. (4.21b), $-\omega(1,0,0,0,1)$, one needs to combine the shuffle relations eqs. (4.22) and (4.23) with eqs. (2.58) and (2.59). The desired result then follows from the constant eMZVs in eq. (4.15) and $\omega(2) = -2\zeta_2$ which is a special case of

$$\omega(n) = \begin{cases} -2\zeta_n & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases}\ . \tag{4.25}$$

The expression for $\omega(n)$ can be inferred from order q^0 in the expansions eqs. (3.35) and (3.36).

4.3.4 Third order in P_{ij}: integrals c_i^3

Starting from the third order in Mandelstam variables, relations such as eq. (4.14) are no longer sufficient to reduce the complete list of c_i^3 in eq. (4.11) to elementary integrals over monomials in P_{12}, P_{13} and P_{14}. Instead, the inevitable factors of P_{23}, P_{24} and P_{34} require the procedure described in eq. (4.24) together with the recursive identities eq. (2.41) to (2.43) in order to rearrange the labels of the eIIs. This allows to reduce integrals over arbitrary monomials in P_{ij} with $1 \leq i < j \leq 4$ to eMZVs. The integrals c_i^3, which are cubic in P_{ij}, give rise to

$$c_1^3 = \omega(1,1,1,0,0,0) \tag{4.26a}$$
$$c_2^3 = \omega(1,1,1,0,0,0) + \omega(1,1,0,1,0,0) + \omega(1,0,1,1,0,0) + \omega(0,1,1,1,0,0) \tag{4.26b}$$
$$c_3^3 = -\omega(1,1,0,0,0,1) \tag{4.26c}$$
$$c_4^3 = 6\omega(1,1,1,0,0,0) + 3\omega(1,1,0,1,0,0) + \omega(1,0,1,1,0,0) + \omega(1,1,0,0,0,1) \tag{4.26d}$$
$$c_5^3 = -\omega(1,1,0,0,0,1) \tag{4.26e}$$
$$c_6^3 = 3\omega(1,1,1,0,0,0) + \omega(1,1,0,1,0,0) \tag{4.26f}$$
$$c_7^3 = 3\omega(1,1,1,0,0,0) + 2\omega(1,1,0,1,0,0) + \omega(1,0,1,1,0,0) \tag{4.26g}$$
$$c_8^3 = 2\omega(2,0,0,0,0,1) + \omega(0,2,0,0,0,1) - 2\omega(1,1,0,0,0,1) - \zeta_2\omega(1,0,0,0) \tag{4.26h}$$
$$\begin{aligned} c_9^3 = {} & 2\omega(2,0,0,0,1,0) + 2\omega(2,0,0,0,0,1) + \omega(0,2,0,0,1,0) + \omega(0,2,0,0,0,1) \\ & - 2\omega(1,1,0,0,1,0) - 2\omega(1,1,0,0,0,1) - \zeta_2\omega(1,0,0,0) - \zeta_2\omega(0,1,0,0) \end{aligned} \tag{4.26i}$$
$$c_{10}^3 = -2\omega(1,1,0,0,0,1) - \omega(1,0,1,0,0,1) \tag{4.26j}$$
$$c_{11}^3 = -2\omega(1,1,0,0,0,1) - \omega(1,0,1,0,0,1) \tag{4.26k}$$
$$\begin{aligned} c_{12}^3 = {} & -2\omega(2,0,0,0,0,1) - \omega(0,2,0,0,0,1) + \zeta_2\omega(1,0,0,0) \\ & - 2\omega(1,0,1,0,0,1) - 2\omega(1,1,0,0,0,1) \ , \end{aligned} \tag{4.26l}$$

where the occurrences of ζ_2 can be traced back to eq. (2.47).

4.3.5 Assembling the results

Momentum conservation only admits particular linear combinations of c_i^j in the four-point amplitude eq. (4.12). It turns out that for all cases considered divergent eMZVs with the singular integrand $f^{(1)}$ in the first or last position drop out. Up to third order in s_{ij}, we have

$$\begin{aligned} I_{4\text{pt}}(1,2,3,4) = {} & \omega(0,0,0) \ - \ 2\omega(0,1,0,0)\,(s_{12}+s_{23}) \ + \ 2\omega(0,1,1,0,0)\,(s_{12}^2+s_{23}^2) \\ & - \ 2\omega(0,1,0,1,0)\,s_{12}s_{23} \ + \ \beta_5\,(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3) \ + \ \beta_{2,3}\,s_{12}s_{23}(s_{12}+s_{23}) \ + \ \mathcal{O}(\alpha'^4) \end{aligned} \tag{4.27}$$

with

$$\beta_5 = \frac{4}{3}\left[\omega(0,0,1,0,0,2) + \omega(0,1,1,0,1,0) - \omega(2,0,1,0,0,0) - \zeta_2\omega(0,1,0,0)\right] \tag{4.28}$$
$$\begin{aligned} \beta_{2,3} = {} & \frac{1}{3}\omega(0,0,1,0,2,0) - \frac{3}{2}\omega(0,1,0,0,0,2) - \frac{1}{2}\omega(0,1,1,1,0,0) \\ & - 2\omega(2,0,1,0,0,0) - \frac{4}{3}\omega(0,0,1,0,0,2) - \frac{10}{3}\zeta_2\omega(0,1,0,0) \ , \end{aligned} \tag{4.29}$$

and the pattern at higher orders is under investigation. The above expressions for β_5 and $\beta_{2,3}$ are obtained using various eMZV relations using the methods of subsection 2.2.3.

4.4 On the q-expansion of eMZVs and the string amplitude

The evaluation of eMZVs as initiated in eq. (4.15) and eq. (4.25) will be pursued systematically in [49, 66, 67]. In this section, we give a glimpse of non-trivial q-dependence in simple cases and provide consistency checks for the constant piece of the low energy expansion eq. (4.27) of the four-point amplitude.

4.4.1 The simplest q-expansions

To determine the q-expansions of the simplest eMZVs, we start from the expansions of $f^{(1)}$ and $f^{(2)}$ spelled out in eq. (3.37), which in turn is based on eqs. (3.32) and (3.33). Using the integrals in appendix C, we arrive at

$$\omega(0,1,0,0) = \frac{\zeta_3}{8\zeta_2} + \frac{3}{2\pi^2} \sum_{m,n=1}^{\infty} \frac{1}{m^3} q^{mn} \tag{4.30}$$

as well as

$$\omega(0,1,1,0,0) = \frac{\zeta_2}{15} - \frac{1}{2\pi^2} \sum_{m,n=1}^{\infty} \frac{n}{m^4} q^{mn} + \frac{1}{3} \sum_{m,n=1}^{\infty} \frac{n}{m^2} q^{mn} \tag{4.31}$$

$$\omega(0,1,0,1,0) = -\frac{\zeta_2}{60} + \frac{2}{\pi^2} \sum_{m,n=1}^{\infty} \frac{n}{m^4} q^{mn} - \frac{1}{3} \sum_{m,n=1}^{\infty} \frac{n}{m^2} q^{mn} \ . \tag{4.32}$$

A systematic method is under investigation and will appear in [67]. Note that the q-dependence of all the examples above can be expressed in terms of the function $\mathrm{ELi}_{n,m}$ introduced in section 8 of ref. [11] at arguments $x = y = 1$.

4.4.2 The constant piece of eMZVs and the α'-derivative

The t-integration in the four-point amplitude eq. (4.1) is divergent unless the choice of gauge group $SO(32)$ leads to cancellations between the cylinder and the Möbius-strip diagram [55]. The divergence is interpreted as a zero-momentum dilaton propagating to the vacuum and therefore proportional to the derivative of the tree level amplitude with respect to α' [58]. The latter is given by

$$A^{\mathrm{tree}}_{\mathrm{string}}(1,2,3,4) = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} A^{\mathrm{tree}}_{\mathrm{YM}}(1,2,3,4) \tag{4.33}$$

with α'-expansion

$$\begin{aligned} \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} &= \exp\left\{ \sum_{k=2}^{\infty} (-1)^k \frac{\zeta_k}{k} \left[s_{12}^k + s_{23}^k - (s_{12}+s_{23})^k \right] \right\} \\ &= 1 - \zeta_2 s_{12}s_{23} + \zeta_3 s_{12}s_{23}(s_{12}+s_{23}) - \zeta_4 s_{12}s_{23}(s_{12}^2 + \tfrac{1}{4}s_{12}s_{23} + s_{23}^2) \\ &\quad - \zeta_5 s_{12}s_{23}(s_{12}^3 + 2s_{12}^2 s_{23} + 2s_{12}s_{23}^2 + s_{23}^3) - \zeta_2\zeta_3 (s_{12}s_{23})^2 (s_{12}+s_{23}) + \mathcal{O}(\alpha'^6) \ . \end{aligned} \tag{4.34}$$

In the representation of the one-loop amplitude given in eq. (4.27), the divergence originates from the constant part of the eMZVs' power series expansion in $q = e^{2\pi i\tau} = e^{-2\pi t}$. A systematic method to extract the constant term of eMZVs will be described in ref. [49]. The resulting divergence in the above result is given by

$$
\begin{aligned}
A^{\text{1-loop}}_{\text{string}}(1,2,3,4)\Big|_{\text{div}} &= s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\, I_{4\text{pt}}(1,2,3,4)\Big|_{q^0} \\
&= \frac{1}{2\pi^2}\, s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\Big\{2\zeta_2 - 3\zeta_3(s_{12}+s_{23}) + 4\zeta_4(s_{12}^2 + \tfrac{1}{4}s_{12}s_{23} + s_{23}^2) \\
&\quad - 5\zeta_5(s_{12}^3 + 2s_{12}^2 s_{23} + 2s_{12}s_{23}^2 + s_{23}^3) + 5\zeta_2\zeta_3 s_{12}s_{23}(s_{12}+s_{23}) + \mathcal{O}(\alpha'^4)\Big\}\ ,
\end{aligned}
\tag{4.35}
$$

which is consistent with the α'-derivative of the tree amplitude [58] upon comparison with eq. (4.34),

$$
A^{\text{1-loop}}_{\text{string}}(1,2,3,4)\Big|_{\text{div}} = -\frac{\alpha'}{2\pi^2}\frac{\partial}{\partial\alpha'}A^{\text{tree}}_{\text{string}}(1,2,3,4)\ . \tag{4.36}
$$

5 Multi-particle one-loop string amplitudes and $f^{(n)}$

This section is devoted to one-loop amplitudes involving five and more open string states. We firstly provide the five-point extension of the four-point α'-expansion in eq. (4.27). It is secondly demonstrated that the doubly-periodic functions $f^{(n)}$ defined in section 3 naturally enter the calculation of one-loop amplitudes with any number of external legs.

5.1 The five-point open string amplitude

In the same way as the four-point open string amplitude in eq. (4.1) allows to factor out the polarization dependence via $A^{\text{tree}}_{\text{YM}}(1,2,3,4)$, one can express the five-point string amplitude in a basis of color-ordered trees of YM theory [68]. BCJ relations [69] single out two independent subamplitudes $A^{\text{tree}}_{\text{YM}}(1,\rho(2,3),4,5)$ with permutation $\rho\in S_2$, and for convenience, we consider the same color orderings in the one-loop string theory counterparts:

$$
A^{\text{1-loop}}_{\text{string}}(1,\sigma(2,3),4,5) = \int_0^\infty \mathrm{d}t \sum_{\rho\in S_2} I_{5\text{pt}}(\sigma|\rho)A^{\text{tree}}_{\text{YM}}(1,\rho(2,3),4,5)\,. \tag{5.1}
$$

The 2×2 matrix $I_{5\text{pt}}(\sigma|\rho)$ is the generalization of the four-point scalar integral $I_{4\text{pt}}(1,2,3,4)$. It can be assembled from the kinematic factors which were simplified in ref. [68] using the pure spinor formalism [70],

$$
\sum_{\rho\in S_2} I_{5\text{pt}}(1|\rho)A^{\text{tree}}_{\text{YM}}(1,\rho(2,3),4,5) = \int_{12345}\prod_{k<l}^{5}\exp\left[s_{kl}P_{kl}\right] \tag{5.2}
$$
$$
\times\left[s_{23}f^{(1)}_{23}\langle C_{1|23,4,5}\rangle + (23\leftrightarrow 24,25,34,35,45)\right]
$$
$$
\langle C_{1|23,4,5}\rangle = s_{45}\left(s_{24}\,A^{\text{tree}}_{\text{YM}}(1,3,2,4,5) - s_{34}A^{\text{tree}}_{\text{YM}}(1,2,3,4,5)\right)\ . \tag{5.3}
$$

The integration measure $\int_{12345}$ is defined in eq. (4.7), the functions $f^{(1)}_{ij} = f^{(1)}(z_i - z_j)$ stem from OPE contractions among the worldsheet fields and the five-point Mandelstam invariants eq. (4.3) can be cast into a five-dimensional basis via momentum conservation, e.g. $s_{13} = s_{45} - s_{12} - s_{23}$.

From the mathematical point of view, the only novel five-point ingredient as compared to the four-point amplitude is the extra factor of $f^{(1)}_{ij} = \partial P_{ij}$ in the integrand of eq. (5.2). Thanks to the embedding of $f^{(1)}$ into the framework of eIIs eq. (2.18), the α'-expansion of the integrals $\int_{12345} f^{(1)}_{ij}\prod_{k<l}^{5}\exp\left[s_{kl}P_{kl}\right]$ in eq. (5.3) is again captured by eMZVs. The detailed discussion of kinematic poles as well as the order-by-order treatment of the exponential will be discussed

elsewhere; here we simply quote the final result:

$$I_{5\text{pt}}(\sigma|\rho) = \big[-\omega(0,0,0)P_2 - 2\omega(0,1,0,0)M_3 - 5\omega(0,1,1,0,0)P_4 \\ - \big(2\omega(0,1,0,1,0) + \tfrac{1}{2}\omega(0,1,1,0,0)\big)L_4 + \mathcal{O}(\alpha'^5)\big]_{\sigma,\rho} \,. \tag{5.4}$$

Up to weight two at order $\mathcal{O}(\alpha^4)$, the eMZV content is the same as in the four-point expansion eq. (4.27). The accompanying 2×2 matrices P_i, M_i, L_i are indexed by permutations ρ, σ, and their entries are polynomials of degree i in Mandelstam variables. The representatives P_i and M_i already appear in the α'-expansion of open-string tree amplitudes, along with even and odd Riemann zeta values ζ_i, respectively [29]. Given that the low-energy limit of one-loop amplitudes at any multiplicity has the mass dimension of $s_{ij}^2 A^{\text{tree}}_{\text{YM}}(\ldots)$, the eMZV coefficients of P_i, M_i, L_i have weight $i-2$. This amounts to a shift of -2 in weight in comparison to the MZV coefficients of P_i, M_i at tree-level.

They are available at the website [39] whereas L_4 reads

$$(L_4)_{11} = s_{12}^2 s_{23}^2 + 2s_{12}^2 s_{23} s_{24} + s_{12}^2 s_{24}^2 + 2s_{12}^2 s_{23} s_{34} + 2s_{12} s_{13} s_{23} s_{34} + 2s_{12} s_{23}^2 s_{34} \\ + 2s_{12}^2 s_{24} s_{34} + s_{12} s_{13} s_{24} s_{34} + 2s_{12} s_{23} s_{24} s_{34} + s_{12}^2 s_{34}^2 + 2s_{12} s_{13} s_{34}^2 \\ + s_{13}^2 s_{34}^2 + 2s_{12} s_{23} s_{34}^2 + 2s_{13} s_{23} s_{34}^2 + s_{23}^2 s_{34}^2 \tag{5.5}$$

$$(L_4)_{12} = -s_{13} s_{24} (3s_{12} s_{23} + s_{13} s_{23} + s_{23}^2 + 2s_{12} s_{24} + s_{13} s_{24} + s_{23} s_{24} \\ + 3s_{12} s_{34} + 2s_{13} s_{34} + 3s_{23} s_{34}) \tag{5.6}$$

and $(L_4)_{22} = (L_4)_{11}\big|_{2\leftrightarrow 3}$ and $(L_4)_{21} = (L_4)_{12}\big|_{2\leftrightarrow 3}$. The relabelling $2 \leftrightarrow 3$ refers to the i, j along with the Mandelstam invariants s_{ij}.

The four-point one-loop amplitude eq. (4.27) can be cast into the same form as eq. (5.4) upon setting $L_4 \to 0$ and

$$P_2 \to -s_{12} s_{23} \,, \quad M_3 \to s_{12} s_{23} \,(s_{12} + s_{23}) \,, \quad P_4 \to -\frac{2}{5}\, s_{12} s_{23} \left(s_{12}^2 + \tfrac{1}{4} s_{12} s_{23} + s_{23}^2 \right) \,, \tag{5.7}$$

in agreement with the four-point open string tree eq. (4.33). The pattern of eMZVs at higher orders in α' as well as the properties of the novel matrices L_i are left for further projects.

5.2 Functions $f^{(n)}$ from the RNS formalism

In this subsection we will show that the doubly-periodic functions $f^{(n)}$ for any n are naturally generated in the one-loop amplitude computation using the RNS formalism [71–73]. Their emergence in the parity-even and parity-odd sectors turns out to follow two separate mechanisms.

5.2.1 Parity-even RNS amplitudes

In the parity-even sector of the RNS computation, the functions $f^{(n)}$ arise from the summation over the even spin structures of the fermions on a genus-one worldsheet. We also take this opportunity to use the method of refs. [74,75] to write down explicit results for the N-point spin sum for $N > 7$.

Definition of $V_p(x_1, \ldots, x_N)$. In the subsequent we use the variables $x_i \equiv z_i - z_{i+1}$ for $i = 1, \ldots, N$ with the condition $z_{N+1} = z_1$ such that $\sum_{i=1}^N x_i = 0$. Using the shorthand $\Omega_i \equiv \alpha\Omega(x_i, \alpha)$ it follows from eq. (3.30) that the α^p-component of $\Omega_1 \cdots \Omega_N$ has at most p

simultaneous single poles in the variables x_i. This suggests the following definition

$$V_p(x_1, x_2, \dots, x_N) \equiv (\Omega_1 \Omega_2 \dots \Omega_N)\Big|_{\alpha^p} . \tag{5.8}$$

For example, with $f_i^{(n)} \equiv f^{(n)}(x_i)$,

$$\begin{aligned}
V_1(x_1, \dots, x_5) &= \sum_{i=1}^{5} f_i^{(1)} \\
V_2(x_1, \dots, x_6) &= \sum_{i=1}^{6} f_i^{(2)} + \sum_{1 \le i < j}^{6} f_i^{(1)} f_j^{(1)} \\
V_3(x_1, \dots, x_7) &= \sum_{i=1}^{7} f_i^{(3)} + \sum_{1 \le i < j}^{7} (f_i^{(1)} f_j^{(2)} + f_i^{(2)} f_j^{(1)}) + \sum_{1 \le i < j < k}^{7} f_i^{(1)} f_j^{(1)} f_k^{(1)} \\
V_4(x_1, \dots, x_8) &= \sum_{i=1}^{8} f_i^{(4)} + \sum_{1 \le i < j}^{8} (f_i^{(1)} f_j^{(3)} + f_i^{(2)} f_j^{(2)} + f_i^{(3)} f_j^{(1)}) + \sum_{1 \le i < j < k < l}^{8} f_i^{(1)} f_j^{(1)} f_k^{(1)} f_l^{(1)} \\
&\quad + \sum_{1 \le i < j < k}^{8} (f_i^{(1)} f_j^{(1)} f_k^{(2)} + f_i^{(1)} f_j^{(2)} f_k^{(1)} + f_i^{(2)} f_j^{(1)} f_k^{(1)}) .
\end{aligned} \tag{5.9}$$

Interestingly, the anti-holomorphic recursion eq. (3.27) implies that $V_p(x_1, \dots, x_N)$ is holomorphic; $\frac{\partial}{\partial \bar{z}_i} V_p(x_1, \dots, x_N) = 0$. Equivalently, the non-holomorphic factors $\mathrm{Im}\,(x_i)$ in $V_p(x_1, \dots, x_N)$ trivially vanish because of the condition $\sum_{i=1}^N x_i = 0$. One can therefore replace $\mathcal{E}_1(x, \tau)$ by $E_1(x, \tau)$ and $f_i^{(n)} \to g_i^{(n)}$ in the notation of subsection 3.3.3 to establish manifest holomorphicity.

Note that the functions in eq. (5.8) were also used in [76] to cast one-loop correlation functions among arbitrary numbers of Kac-Moody currents into a closed form.

Spin sums in one-loop amplitudes. In the computation of parity-even one-loop amplitudes in the RNS formalism the bosonic worldsheet fields can be straightforwardly integrated out to yield products of $f^{(1)}$, possibly after integration by parts. Worldsheet fermions, on the other hand, give rise to the following spin sums,

$$\mathcal{G}_N(x_1, \dots, x_N) \equiv \sum_{\nu=1,2,3} (-1)^\nu \left(\frac{\theta_{\nu+1}(0, \tau)}{\theta_1'(0, \tau)} \right)^4 S_\nu(x_1) S_\nu(x_2) \dots S_\nu(x_N) , \tag{5.10}$$

where $\sum_{i=1}^N x_i = 0$, S_ν is the Szegö kernel and ν denotes the even spin structure with associated Jacobi theta functions $\theta_2, \theta_3, \theta_4$ [77–79, 53],

$$S_\nu(z) \equiv \frac{\theta_1'(0, \tau) \theta_{\nu+1}(z, \tau)}{\theta_{\nu+1}(0, \tau) \theta_1(z, \tau)} . \tag{5.11}$$

A method to evaluate such sums was presented in ref. [74] and its explicit results at $N \le 7$ can be written in terms of $f^{(1)}(z)$, the Weierstrass function $\wp(z)$ and its derivatives $\partial^k \wp(z)$,

$$\begin{aligned}
\mathcal{G}_4(x_1, \dots, x_4) &= 1 \\
\mathcal{G}_5(x_1, \dots, x_5) &= \sum_{j=1}^{5} f^{(1)}(x_j)
\end{aligned}$$

$$\mathcal{G}_6(x_1,\dots,x_6) = \frac{1}{2}\Big\{\Big(\sum_{j=1}^{6} f^{(1)}(x_j)\Big)^2 - \sum_{j=1}^{6}\wp(x_j)\Big\}$$
$$\mathcal{G}_7(x_1,\dots,x_7) = \frac{1}{6}\Big\{\Big(\sum_{j=1}^{7} f^{(1)}(x_j)\Big)^3 - \sum_{j=1}^{7}\partial\wp(x_j) - 3\Big(\sum_{j=1}^{7} f^{(1)}(x_j)\Big)\Big(\sum_{j=1}^{7}\wp(x_j)\Big)\Big\}\,. \quad (5.12)$$

One can show that the above results are naturally described by the elliptic functions $V_p(x_1,\dots,x_N)$,

$$\mathcal{G}_N(x_1,\dots,x_N) = V_{N-4}(x_1,\dots,x_N), \qquad 4 \le N \le 7\,. \quad (5.13)$$

An alternative method was used in [80, 81] to express $\mathcal{G}_N$ in terms of single derivatives of the bosonic Green function. The equivalence of the expression for $\mathcal{G}_6$ given in these references with eq. (5.12) can be verified through the Fay identity eq. (2.39).

Although the results for $N \ge 8$ were not written down explicitly in ref. [74], they also take a natural form when expressed in terms of elliptic functions $V_p(x_1,\dots,x_N)$,

$$\begin{aligned}
\mathcal{G}_8(x_1,\dots,x_8) &= V_4(x_1,\dots,x_8) + 3e_4\\
\mathcal{G}_9(x_1,\dots,x_9) &= V_5(x_1,\dots,x_9) + 3e_4V_1(x_1,\dots,x_9)\\
\mathcal{G}_{10}(x_1,\dots,x_{10}) &= V_6(x_1,\dots,x_{10}) + 3e_4V_2(x_1,\dots,x_{10}) + 10e_6\\
\mathcal{G}_{11}(x_1,\dots,x_{11}) &= V_7(x_1,\dots,x_{11}) + 3e_4V_3(x_1,\dots,x_{11}) + 10e_6V_1(x_1,\dots,x_{11})\\
\mathcal{G}_{12}(x_1,\dots,x_{12}) &= V_8(x_1,\dots,x_{12}) + 3e_4V_4(x_1,\dots,x_{12}) + 10e_6V_2(x_1,\dots,x_{12}) + 42e_8\,. \quad (5.14)
\end{aligned}$$

The factors of the Eisenstein series e_j eq. (3.9) can be systematically computed as well. Following ref. [75], we define $Q_0(\wp) = 1$, $Q_1(\wp) = \wp$ and $Q_{k+1}(\wp) = \wp^{(2k)}$. For example,

$$\begin{aligned}
Q_2(\wp) &= 3!\wp^2 - \tfrac{1}{2}g_2\\
Q_3(\wp) &= 5!\wp^3 - 18g_2\wp - 12g_3\\
Q_4(\wp) &= 7!\wp^4 - 1008g_2\wp^2 - 720g_3\wp + 9g_2^2\\
Q_5(\wp) &= 9!\wp^5 - 90720g_2\wp^3 - 64800g_3\wp^2 + 3024g_2^2\wp + 2376g_2g_3\,, \quad (5.15)
\end{aligned}$$

where the Weierstrass equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ has been used to rewrite the $2k^{\text{th}}$ derivative of $\wp$ as a polynomial in $\wp$. In the above equation, $g_2 = -4(s_1s_2 + s_2s_3 + s_3s_1) = 60e_4$, $g_3 = 4s_1s_2s_3 = 140e_6$ are the elliptic invariants and s_i are the branch points of the genus-one elliptic curve $y^2 = 4(z-s_1)(z-s_2)(z-s_3)$ satisfying $s_1+s_2+s_3 = 0$. Defining

$$F_{2k-4} \equiv -\frac{1}{(2k-1)!}\frac{\big[(s_1-s_3)Q_k(s_2) + (s_3-s_2)Q_k(s_1) + (s_2-s_1)Q_k(s_3)\big]}{(s_1-s_3)(s_3-s_2)(s_2-s_1)}, \quad k \ge 4 \quad (5.16)$$

straightforward calculation leads to[11]

$$F_4 = 3e_4, \quad F_6 = 10e_6, \quad F_8 = 42e_8, \quad F_{10} = 168e_{10}, \quad F_{12} = 627e_{12} + 9e_4^3 \,,$$

which precisely captures the factors of e_j in eq. (5.14). We have explicitly checked up to $N = 12$ that the spin sums can be uniformly written as,

$$\mathcal{G}_N(x_1, \ldots, x_N) = V_{N-4}(x_1, \ldots, x_N) + \sum_{k=1}^{\lfloor \frac{N-8}{2} \rfloor + 1} F_{2k+2} V_{N-2k-6}(x_1, \ldots, x_N)\,. \tag{5.17}$$

5.2.2 Parity-odd RNS amplitudes

The parity-odd sector of the RNS computation entirely stems from the unique odd spin structure at genus one where the worldsheet spinors obey anti-periodic boundary conditions along both torus cycles and acquire a zero mode. The worldsheet integrand is governed by zero-mode saturation and, probably as a common feature with the Green-Schwarz or pure spinor formalism, OPE contractions of the worldsheet fields which generate $N-4$ factors of $f^{(1)}_{ij}$ where $f^{(n)}_{ij} \equiv f^{(n)}(z_i - z_j)$.

For six points, the direct evaluation of the OPEs gives rise to a quadratic factor $f^{(1)}_{ij} f^{(1)}_{kl}$ for various combinations of labels capturing the behavior of the singularities as the vertices collide. However, we know from the Fay identity eq. (2.39) that these quadratic combinations are not linearly independent and therefore one is naturally led to higher-weight $f^{(n)}$'s when considering a minimal basis of integrals to evaluate. The simplest example where a higher-weight $f^{(n)}$ is generated this way is $f^{(1)}_{12} f^{(1)}_{13} + f^{(1)}_{23} f^{(1)}_{21} + f^{(1)}_{31} f^{(1)}_{32} = f^{(2)}_{12} + f^{(2)}_{23} + f^{(2)}_{31}$ which can be viewed as generalizing the genus-zero partial fraction identity eq. (2.38). The non-vanishing of the right-hand side provides an important distinction between one-loop and tree-level string amplitudes and it is ultimately related to the gauge anomaly cancellation mechanism in the superstring [56, 57]. It can be shown that the parity-odd part of the six-point amplitude as firstly computed in ref. [82] can be entirely written in terms of $f^{(2)}$, i.e. that any appearance of $f^{(1)}$ can be removed via eq. (2.39).

More generally, the $N-4$ powers of $f^{(1)}$ in the N-point amplitude allow, via the Fay identity, the generation of $f^{(p)}$ with up to $p = N-4$. In this way the need for a general integration method for the type of iterated integrals on an elliptic curve considered in this paper is justified.

6 Discussion and further directions

In this article, we have proposed an organization scheme for elliptic iterated integrals and elliptic multiple zeta values (eMZVs), where the key definitions are provided in eqs. (2.18) and (2.22). The infinite family of doubly-periodic functions $f^{(n)}$ appearing in the integrands of section 2 are put into a mathematical context and are related to multiple elliptic polylogarithms in section 3.

[11] The Eisenstein series e_8, e_{10} and e_{12} can be written in terms of e_4 and e_6 as follows

$$e_8 = \frac{3}{7} e_4^2, \quad e_{10} = \frac{5}{11} e_4 e_6, \quad e_{12} = \frac{18}{143} e_4^3 + \frac{25}{143} e_6^2.$$

The general formula is written in terms of $d_k \equiv (2k+3)k! e_{2k+4}$

$$d_{n+2} = \frac{3n+6}{2n+9} \sum_{k=0}^{n} \binom{n}{k} d_k d_{n-k}\,.$$

As a first natural and simple application of this framework, we have identified eMZVs in the α'-expansion of one-loop scattering amplitudes in open string theory. The leading orders in the low-energy behavior of the four- and five-point amplitudes in terms of eMZVs are presented in eqs. (4.27) and (5.4). Divergent eMZVs turn out to cancel from our results.

Having demonstrated the potential of the formalism for an initial example, there are numerous open questions to be pursued in the near future. Most obviously, the eMZV content of the low energy expansion of cylinder amplitudes needs to be understood for higher orders in α', which can be done conveniently using the new techniques. Furthermore, the contributions from the cylinder configuration with open string insertions on both boundaries as well as from the Möbius-strip topology shall be determined in terms of the iterated integrals introduced in subsection 2.2. The q-expansion of eMZVs exemplified in section 4.4.1 offers a promising approach to systematically perform the t-integration in eq. (4.1) after summing all topologies for the gauge group $SO(32)$ [55].

On the mathematical side, the network of relations between eMZVs explored in subsection 2.2.3 will be further investigated in refs. [49, 66, 67]. A suitable coaction along the lines of refs. [5–8, 16] might lead to a natural basis choice for eMZVs and might allow to further identify patterns in the one-loop string amplitudes. In the same way as the Drinfeld associator was instrumental in understanding the pattern of MZVs [29] in open string tree-level amplitudes [37] and finally allowed to completely determine their α'-expansion in ref. [38], the elliptic associators discussed in ref. [24] might encode the structure of the α'-expansion at one-loop. Furthermore, in refs. [83, 84] so-called *multiple modular values* are discussed whose possible relation to the eMZVs studied here needs to be explored.

In multi-particle one-loop open string amplitudes, the pure spinor formalism, in particular the ingredients of ref. [85] are expected to yield a compact description of the kinematic factors associated to the functions $f^{(n)}$. While the precise superspace kinematic factors along with various powers of $f^{(1)}$ have been derived in ref. [68], the kinematic companions of $f^{(n\geq 2)}$ in the higher-point amplitudes are currently under investigation.

Finally, it would be desirable to find a similar scheme for organizing the α'-expansion of closed string one-loop amplitudes. In particular, the worldsheet integrals investigated in refs. [63–65] might allow for a description in terms of eMZVs and their counterpart defined with respect to the other cycle of the torus. The peculiar linear combinations of torus integrals appearing in the α'-expansion of closed-string amplitudes call for an explanation along the lines of the above finding that divergent eMZVs drop out from the open-string expansions.

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Appendix

A Derivatives of multiple polylogarithms w.r.t. the labels

The proof of the recursion in eq. (2.11) relies on the derivatives of multiple polylogarithms eq. (2.1) with respect to their labels $a_1, a_2, \ldots, a_n$ [5]:

$$\frac{\partial}{\partial z} G(\vec{a}; z) = \frac{1}{z - a_1} G(a_2, \ldots, a_n; z). \tag{A.1}$$

$$\frac{\partial}{\partial a_i} G(\vec{a}; z) = \frac{1}{a_{i-1} - a_i} G(\ldots, \hat{a}_{i-1}, \ldots; z) \; + \; \frac{1}{a_i - a_{i+1}} G(\ldots, \hat{a}_{i+1}, \ldots; z)$$
$$- \frac{a_{i-1} - a_{i+1}}{(a_{i-1} - a_i)(a_i - a_{i+1})} G(\ldots, \hat{a}_i, \ldots; z) \;, \quad i \neq 1, n \tag{A.2}$$

$$\frac{\partial}{\partial a_n} G(\vec{a}; z) = \frac{1}{a_{n-1} - a_n} G(\ldots, \hat{a}_{n-1}, a_n; z) - \frac{a_{n-1}}{(a_{n-1} - a_n) a_n} G(\ldots, a_{n-1}; z) \;. \tag{A.3}$$

B Identities for iterated integrals

This appendix provides further relations to integrate eIIs whose argument occurs in the labels.

B.1 Total derivatives

The following identities generalize eqns. (2.34) to (2.36) for multiple successive occurrences of the argument t_0 in the label. If the first k labels match the argument, one can show that

$$\frac{\mathrm{d}}{\mathrm{d}t_0} \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_k & n_{k+1} & \ldots & n_r \\ t_0 & t_0 & \ldots & t_0 & a_{k+1} & \ldots & a_r \end{smallmatrix}; t_0 \right) \tag{B.1}$$
$$= \left(\prod_{j=1}^{k-1} \int_0^{t_{j-1}} \mathrm{d}t_j \, f^{(n_j)}(t_j - t_0) \right) \int_0^{t_{k-1}} \mathrm{d}t \, f^{(n_k)}(t - t_0) f^{(n_{k+1})}(t - a_{k+1}) \, \Gamma\left(\begin{smallmatrix} n_{k+2} & \ldots & n_r \\ a_{k+2} & \ldots & a_r \end{smallmatrix}; t \right) \;.$$

For a terminal sequence of $a_j = t_0$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t_0} \Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{\ell-1} & n_\ell & \ldots & n_r \\ a_1 & \ldots & a_{\ell-1} & t_0 & \ldots & t_0 \end{smallmatrix}; t_0 \right) = f^{(n_1)}(t_0 - a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_{\ell-1} & n_\ell & \ldots & n_r \\ a_2 & \ldots & a_{\ell-1} & t_0 & \ldots & t_0 \end{smallmatrix}; t_0 \right)$$
$$- \left(\prod_{j=1}^{\ell-2} \int_0^{t_{j-1}} \mathrm{d}t_j \, f^{(n_j)}(t_j - a_j) \right) \int_0^{t_{\ell-2}} \mathrm{d}t \, f^{(n_{\ell-1})}(t - a_{\ell-1}) f^{(n_\ell)}(t - t_0) \, \Gamma\left(\begin{smallmatrix} n_{\ell+1} & \ldots & n_r \\ t_0 & \ldots & t_0 \end{smallmatrix}; t \right)$$
$$+ f^{(n_r)}(-t_0) \, \Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{\ell-1} & n_\ell & \ldots & n_{r-1} \\ a_1 & \ldots & a_{\ell-1} & t_0 & \ldots & t_0 \end{smallmatrix}; t_0 \right) \;. \tag{B.2}$$

Finally, an intermediate sequence of $a_j = t_0$ ranging from $j = p$ to $j = q$ with $p \neq 1$ and $q \neq r$ can be addressed via

$$\frac{\mathrm{d}}{\mathrm{d}t_0} \Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{p-1} & n_p & \ldots & n_q & n_{q+1} & \ldots & n_r \\ a_1 & \ldots & a_{p-1} & t_0 & \ldots & t_0 & a_{q+1} & \ldots & a_r \end{smallmatrix}; t_0 \right) = f^{(n_1)}(t_0 - a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_{p-1} & n_p & \ldots & n_q & n_{q+1} & \ldots & n_r \\ a_2 & \ldots & a_{p-1} & t_0 & \ldots & t_0 & a_{q+1} & \ldots & a_r \end{smallmatrix}; t_0 \right)$$
$$- \left(\prod_{j=1}^{p-2} \int_0^{t_{j-1}} \mathrm{d}t_j \, f^{(n_j)}(t_j - a_j) \right)$$
$$\times \int_0^{t_{p-2}} \mathrm{d}t \, f^{(n_{p-1})}(t - a_{p-1}) f^{(n_p)}(t - t_0) \, \Gamma\left(\begin{smallmatrix} n_{p+1} & \ldots & n_q & n_{q+1} & \ldots & n_r \\ t_0 & \ldots & t_0 & a_{q+1} & \ldots & a_r \end{smallmatrix}; t \right)$$

$$+\left(\prod_{j=1}^{p-1}\int_0^{t_{j-1}}\mathrm{d}t_j\,f^{(n_j)}(t_j-a_j)\right)\left(\prod_{j=p}^{q-1}\int_0^{t_{j-1}}\mathrm{d}t_j\,f^{(n_j)}(t_j-t_0)\right) \quad \text{(B.3)}$$
$$\times\int_0^{t_{q-1}}\mathrm{d}t\,f^{(n_q)}(t-t_0)f^{(n_{q+1})}(t-a_{q+1})\,\Gamma\left(\begin{smallmatrix}n_{q+2}&\dots&n_r\\a_{q+2}&\dots&a_r\end{smallmatrix};t\right)\ .$$

Cases with multiple disconnected sequences of $a_j=t_0$ can be treated along similar lines.

B.2 Recursive removal of the argument from the labels

On the basis of eqns. (B.1) to (B.3), we can generalize the recursions eqns. (2.41) to (2.43) to situations where several successive instances of the argument occur among the labels. If the first k labels match the argument, one can show that

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix}n_1&n_2&\dots&n_k&n_{k+1}&\dots&n_r\\z&z&\dots&z&a_{k+1}&\dots&a_r\end{smallmatrix};z\right)&=\lim_{z\to0}G(z,\dots,z,a_{k+1},\dots,a_r;z)\prod_{j=1}^{r}\delta_{n_j,1}\\
&-(-1)^{n_k}\int_0^z\mathrm{d}t\,f^{(n_k+n_{k+1})}(t-a_{k+1})\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{k-1}&0&n_{k+2}&\dots&n_r\\t&\dots&t&0&a_{k+2}&\dots&a_r\end{smallmatrix};t\right)\\
&+\sum_{j=0}^{n_{k+1}}\binom{n_k-1+j}{j}\int_0^z\mathrm{d}t\,f^{(n_{k+1}-j)}(t-a_{k+1})\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{k-1}&n_k+j&n_{k+2}&\dots&n_r\\t&\dots&t&t&a_{k+2}&\dots&a_r\end{smallmatrix};t\right)\\
&+\sum_{j=0}^{n_k}\binom{n_{k+1}-1+j}{j}(-1)^{n_k+j}\int_0^z\mathrm{d}t\,f^{(n_k-j)}(t-a_{k+1})\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{k-1}&n_{k+1}+j&n_{k+2}&\dots&n_r\\t&\dots&t&a_{k+1}&a_{k+2}&\dots&a_r\end{smallmatrix};t\right)\ .
\end{aligned} \quad \text{(B.4)}$$

For a terminal sequence of $a_j=z$, we find

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{\ell-1}&n_\ell&\dots&n_r\\a_1&\dots&a_{\ell-1}&z&\dots&z\end{smallmatrix};z\right)&=\lim_{z\to0}G(a_1,\dots,a_{\ell-1},z,\dots,z;z)\prod_{j=1}^{r}\delta_{n_j,1}\\
&+\int_0^z\mathrm{d}t\,f^{(n_1)}(t-a_1)\,\Gamma\left(\begin{smallmatrix}n_2&\dots&n_{\ell-1}&n_\ell&\dots&n_r\\a_2&\dots&a_{\ell-1}&t&\dots&t\end{smallmatrix};t\right)\\
&+(-1)^{n_\ell}\int_0^z\mathrm{d}t\,f^{(n_\ell+n_{\ell-1})}(t-a_{\ell-1})\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{\ell-2}&0&n_{\ell+1}&\dots&n_r\\a_1&\dots&a_{\ell-2}&0&t&\dots&t\end{smallmatrix};t\right)\\
&-\sum_{j=0}^{n_{\ell-1}}\binom{n_\ell-1+j}{j}\int_0^z\mathrm{d}t\,f^{(n_{\ell-1}-j)}(t-a_{\ell-1})\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{\ell-2}&n_\ell+j&n_{\ell+1}&\dots&n_r\\a_1&\dots&a_{\ell-2}&t&t&\dots&t\end{smallmatrix};t\right)\\
&-\sum_{j=0}^{n_\ell}\binom{n_{\ell-1}-1+j}{j}(-1)^{n_\ell+j}\int_0^z\mathrm{d}t\,f^{(n_\ell-j)}(t-a_{\ell-1})\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{\ell-2}&n_{\ell-1}+j&n_{\ell+1}&\dots&n_r\\a_1&\dots&a_{\ell-2}&a_{\ell-1}&t&\dots&t\end{smallmatrix};t\right)\\
&+(-1)^{n_r}\int_0^z\mathrm{d}t\,f^{(n_r)}(t)\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{\ell-1}&n_\ell&\dots&n_{r-1}\\a_1&\dots&a_{\ell-1}&t&\dots&t\end{smallmatrix};t\right)\ .
\end{aligned} \quad \text{(B.5)}$$

Finally, an intermediate sequence of $a_j=z$ ranging from $j=p$ to $j=q$ with $p\neq1$ and $q\neq r$ can be addressed via

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{p-1}&n_p&\dots&n_q&n_{q+1}&\dots&n_r\\a_1&\dots&a_{p-1}&z&\dots&z&a_{q+1}&\dots&a_r\end{smallmatrix};z\right)&=\lim_{z\to0}G(a_1,\dots,a_{p-1},z,\dots,z,a_{q+1},\dots,a_r;z)\prod_{j=1}^{r}\delta_{n_j,1}\\
&+\int_0^z\mathrm{d}t\,f^{(n_1)}(t-a_1)\,\Gamma\left(\begin{smallmatrix}n_2&\dots&n_{p-1}&n_p&\dots&n_q&n_{q+1}&\dots&n_r\\a_2&\dots&a_{p-1}&t&\dots&t&a_{q+1}&\dots&a_r\end{smallmatrix};t\right)\\
&+(-1)^{n_p}\int_0^z\mathrm{d}t\,f^{(n_p+n_{p-1})}(t-a_{p-1})\,\Gamma\left(\begin{smallmatrix}n_1&\dots&n_{p-2}&0&n_{p+1}&\dots&n_q&n_{q+1}&\dots&n_r\\a_1&\dots&a_{p-2}&0&t&\dots&t&a_{q+1}&\dots&a_r\end{smallmatrix};t\right)
\end{aligned} \quad \text{(B.6)}$$

$$
\begin{aligned}
&- \sum_{j=0}^{n_{p-1}} \binom{n_p-1+j}{j} \int_0^z \mathrm{d}t\, f^{(n_{p-1}-j)}(t-a_{p-1})\, \Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{p-2} & n_p+j & n_{p+1} & \dots & n_q & n_{q+1} & \dots & n_r \\ a_1 & \dots & a_{p-2} & t & t & \dots & t & a_{q+1} & \dots & a_r \end{smallmatrix} ;t\right) \\
&- \sum_{j=0}^{n_p} \binom{n_{p-1}-1+j}{j} (-1)^{n_p+j} \int_0^z \mathrm{d}t\, f^{(n_p-j)}(t-a_{p-1})\, \Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{p-2} & n_{p-1}+j & n_{p+1} & \dots & n_q & n_{q+1} & \dots & n_r \\ a_1 & \dots & a_{p-2} & a_{p-1} & t & \dots & t & a_{q+1} & \dots & a_r \end{smallmatrix} ;t\right) \\
&- (-1)^{n_q} \int_0^z \mathrm{d}t\, f^{(n_q+n_{q+1})}(t-a_{q+1})\, \Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{p-1} & n_p & \dots & n_{q-1} & 0 & n_{q+2} & \dots & n_r \\ a_1 & \dots & a_{p-1} & t & \dots & t & 0 & a_{q+2} & \dots & a_r \end{smallmatrix} ;t\right) \\
&+ \sum_{j=0}^{n_{q+1}} \binom{n_q-1+j}{j} \int_0^z \mathrm{d}t\, f^{(n_{q+1}-j)}(t-a_{q+1})\, \Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{p-1} & n_p & \dots & n_{q-1} & n_q+j & n_{q+2} & \dots & n_r \\ a_1 & \dots & a_{p-1} & t & \dots & t & t & a_{q+2} & \dots & a_r \end{smallmatrix} ;t\right) \\
&+ \sum_{j=0}^{n_q} \binom{n_{q+1}-1+j}{j} (-1)^{n_q+j} \int_0^z \mathrm{d}t\, f^{(n_q-j)}(t-a_{q+1})\, \Gamma\left(\begin{smallmatrix} n_1 & \dots & n_{p-1} & n_p & \dots & n_{q-1} & n_{q+1}+j & n_{q+2} & \dots & n_r \\ a_1 & \dots & a_{p-1} & t & \dots & t & a_{q+1} & a_{q+2} & \dots & a_r \end{smallmatrix} ;t\right) \ .
\end{aligned}
$$

These relations reproduce eqns. (2.41) to (2.43) for $k=1$, $p=q$ and $\ell=r$, respectively.

B.3 Eliminating labels $a_j = z$ at length three

The generalization of eq. (2.52) to length three is governed by

$$
\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_1 & n_2 & n_3 \\ z & 0 & 0 \end{smallmatrix} ;z\right) &= -\zeta_3 \delta^1_{n_1}\delta^1_{n_2}\delta^1_{n_3} + \zeta_2 \sum_{j=0}^{n_2} \delta^1_{n_3}\delta^1_{n_1+j} \binom{n_1-1+j}{j} \Gamma(n_2-j;z) \\
&- (-1)^{n_1}\Gamma(n_1+n_2,0,n_3;z) + \sum_{j=0}^{n_1} (-1)^{n_1+j} \binom{n_2-1+j}{j} \Gamma(n_1-j,n_2+j,n_3;z) \\
&- \sum_{j=0}^{n_2} (-1)^{n_1+j} \binom{n_1-1+j}{j} \Gamma(n_2-j,n_1+n_3+j,0;z) \\
&+ \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} \sum_{k=0}^{n_3} (-1)^{n_1+j+k} \binom{n_1+j-1+k}{k} \Gamma(n_2-j,n_3-k,n_1+j+k;z) \\
&+ \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} \sum_{k=0}^{n_1+j} (-1)^{n_1+j+k} \binom{n_3-1+k}{k} \Gamma(n_2-j,n_1+j-k,n_3+k;z) \ .
\end{aligned}
\tag{B.7}
$$

The reflection identity (2.20) allows to infer $\Gamma\left(\begin{smallmatrix} n_1 & n_2 & n_3 \\ z & z & 0 \end{smallmatrix} ;z\right) = (-1)^{n_1+n_2+n_3}\Gamma\left(\begin{smallmatrix} n_3 & n_2 & n_1 \\ z & 0 & 0 \end{smallmatrix} ;z\right)$, and permutations in the labels are covered by shuffle relations.

C Trigonometric integrals

This appendix gathers trigonometric integrals relevant for the evaluation of eMZVs. The result in eq. (4.30) for $\omega(0,1,0,0)$ relies on

$$
\int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2\, \sin(2\pi n z_2)\, z_2 = \frac{3}{8\pi^3 n^3} \tag{C.1}
$$

$$
\int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2\, \cot(\pi z_2)\, z_2 = \frac{3\zeta_3}{4\pi^3} \ , \tag{C.2}
$$

and the eMZVs relevant at order s_{ij}^2 as given by eq. (4.31) and eq. (4.32) are based on

$$
\int_0^1 \mathrm{d}z_5 \int_0^{z_5} \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \int_0^{z_2} \mathrm{d}z_1\, \cos(2\pi n z_1) = \frac{1}{24\pi^2 n^2} - \frac{1}{16\pi^4 n^4} \tag{C.3}
$$

$$\int_0^1 \mathrm{d}z_5 \int_0^{z_5} \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \, \cos(2\pi n z_2)\, z_2 = -\frac{1}{24\pi^2 n^2} + \frac{1}{4\pi^4 n^4} \; . \tag{C.4}$$

D Cycle index of the symmetric group and the $f^{(n)}$ functions

This appendix highlights the connection between the explicit expansion of the doubly-periodic functions $f^{(n)}$ in (3.23) with the cycle index of the symmetric group S_n. For general references, see [86, 87].

Cycle structures. Every permutation $g \in S_n$ of $X = \{1, \ldots, n\}$ can be written as the product of disjoint cycles with lengths $a_1, \ldots, a_n$ such that $n = \sum_{i=1}^n a_i$. This integer partition of n is represented by $\lambda = 1^{a_1} 2^{a_2} \ldots n^{a_n}$ and is called the cycle structure of the permutation. Therefore the total number of cycle structures for the permutations in S_n is given by the integer partition $P(n) = 1, 2, 3, 5, 7, \ldots$. Note that the number of terms in each $f^{(n)}$ is also $P(n)$. Furthermore, if $\lambda = 1^{a_1} 2^{a_2} \ldots n^{a_n}$ is a partition of n (denoted by $\lambda \vdash n$), the number of permutations with cycle structure λ is [86]

$$\frac{n!}{\prod_{i=1}^n i^{a_i} a_i!} \; . \tag{D.1}$$

Note that the coefficients of the monomials $\mathcal{E}_1^{a_1} \ldots \mathcal{E}_n^{a_n}$ in $f^{(n)}$ given by eq. (3.24) are reproduced by the formula (D.1) with the corresponding cycle structure. This observation can be made more precise with the definition of the *cycle index* of the symmetric group S_n [86],

$$Z(S_n; s_1, \ldots, s_n) = \frac{1}{n!} \sum_{g \in S_n} z(g; s_1, \ldots, s_n) \; , \tag{D.2}$$

where $z(g; s_1, \ldots, s_n) = s_1^{a_1} s_2^{a_2} \ldots s_n^{a_n}$ and a_i counts the number of cycles of length i in the permutation g. One can see from the first few examples[12],

$$\begin{aligned}
Z(S_1, s_1) &= s_1 \\
Z(S_2, s_1, \ldots, s_2) &= \frac{1}{2!}(s_1^2 + s_2) \\
Z(S_3, s_1, \ldots, s_3) &= \frac{1}{3!}(s_1^3 + 3 s_1 s_2 + 2 s_3) \\
Z(S_4, s_1, \ldots, s_4) &= \frac{1}{4!}(s_1^4 + 6 s_1^2 s_2 + 8 s_1 s_3 + 3 s_2^2 + 6 s_4)
\end{aligned}$$

that the cycle index of S_n captures the expansions in (3.23). More precisely, theorem 1.3.3 of [88] can be written as

$$\sum_{n=0}^{\infty} \alpha^n Z(S_n; \mathcal{E}_1, \ldots, \mathcal{E}_n) = \exp\Big(\sum_{j=1}^{\infty} \frac{\mathcal{E}_j}{j} \alpha^j \Big) \; , \tag{D.3}$$

and comparing (3.22) with (D.3) leads to,

$$\begin{aligned}
f^{(n)} &= Z(S_n; \mathcal{E}_1, \ldots, \mathcal{E}_n) \\
&= \sum_{\lambda \vdash n} \prod_{i=1}^{n} \frac{\mathcal{E}_i^{a_i}}{i^{a_i} a_i!}, \quad \lambda = 1^{a_1} 2^{a_2} \ldots n^{a_n} \; .
\end{aligned} \tag{D.4}$$

[12] In addition, it is convenient to define $Z(S_0) = 1$.

Furthermore, one can also show that [87],

$$\frac{\partial f^{(n)}(z,\tau)}{\partial \mathcal{E}_p} = \frac{1}{p} f^{(n-p)}(z,\tau). \tag{D.5}$$

Note, in particular, that (D.5) yields an alternative proof of (3.27),

$$\frac{\partial f^{(n)}(z,\tau)}{\partial \bar{z}} = \frac{\partial f^{(n)}(z,\tau)}{\partial \mathcal{E}_1} \frac{\partial \mathcal{E}_1}{\partial \bar{z}} = -\frac{\pi}{\mathrm{Im}(\tau)} f^{(n-1)}(z,\tau). \tag{D.6}$$

Symmetric polynomials. The cycle index of the symmetric group S_n also provides a recipe for expressing the *complete symmetric function* h_j in terms of the *power sum function* p_j, i.e., $h_n = Z(S_n; p_1, p_2, \ldots, p_n)$ [86]. Therefore the functional form of h_n matches that of $f^{(n)}$ and one can exploit the well-known relation $nh_n = \sum_{i=1}^{n} p_i h_{n-i}$ from the theory of symmetric functions to obtain the corresponding recursion formula eq. (3.25) for $f^{(n)}$.

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Appendix D

Relations between elliptic multiple zeta values and a special derivation algebra

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Relations between elliptic multiple zeta values and a special derivation algebra

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Abstract

We investigate relations between elliptic multiple zeta values and describe a method to derive the number of indecomposable elements of given weight and length. Our method is based on representing elliptic multiple zeta values as iterated integrals over Eisenstein series and exploiting the connection with a special derivation algebra. Its commutator relations give rise to constraints on the iterated integrals over Eisenstein series relevant for elliptic multiple zeta values and thereby allow to count the indecomposable representatives. Conversely, the above connection suggests apparently new relations in the derivation algebra. Under `https://tools.aei.mpg.de/emzv` we provide relations for elliptic multiple zeta values over a wide range of weights and lengths.

Contents

1 Introduction

While multiple zeta values (MZVs) have been a very active field of research during the last decades, their elliptic analogues have received more attention only recently. Pioneered by the work of Enriquez [1], Levin [2], Levin and Racinet [3] as well as Brown and Levin [4], many properties of elliptic multiple zeta values (eMZVs) have been identified. While mathematically interesting objects in their own right, eMZVs, the associated elliptic iterated integrals as well as related objects such as multiple elliptic polylogarithms appear in various contexts in quantum field theory and string theory. Well-known examples include the one-loop amplitude in open superstring theory [5] as well as the sunset Feynman integral and its generalizations [6–9]. We would also like to mention a recent application of elliptic functions to the reduction of Feynman integrals using maximal unitarity cuts [10].

Algebraic relations between usual MZVs are well understood based on their conjectural structure as a Hopf algebra comodule [11, 12]. The number of indecomposable MZVs of given weight and depth is expected to be given by the Broadhurst-Kreimer conjecture [13], which is in line with Zagier's conjecture [14] on the counting of MZVs at fixed weight. A comprehensive collection of relations among MZVs has been made available in the MZV data mine [15].

In this article, we investigate relations between eMZVs and classify their *indecomposable* representatives. Apart from the shuffle relations which are immediately implied by their definition as iterated integrals, eMZVs are related by Fay identities. Those identities are the generalization of partial-fraction identities, which appear in the context of usual MZVs. Both shuffle and Fay relations preserve the overall modular weight of the integrand which appears to furnish a natural analogue of the MZVs' transcendental weight. We will describe a systematic way of exploiting the combination of the two types of relations. However, the application of this method to higher weights and lengths suffers from the proliferating combinatorics of the Fay relations.

An alternative and computationally more efficient way of studying relations between eMZVs consists of employing their Fourier expansion in $q = e^{2\pi i\tau}$, where τ is the modular parameter of the elliptic curve. The q-derivative of eMZVs is known from ref. [1] in terms of Eisenstein series and eMZVs of lower length. Since eMZVs degenerate to MZVs at the cusp $q \to 0$ in a manner described in refs. [1, 16], the supplementing boundary value is available as well. Hence, the differential equation can be integrated to yield the q-expansion of eMZVs recursively to – in principle – arbitrarily high order. Once the q-expansion of an eMZV is available up to a certain power in q, finding relations between eMZVs valid up to this particular power and identifying indecomposable representatives amounts to solving a linear system.

Clearly, the agreement of the respective q-expansions up to a certain power in q is necessary but not sufficient for the validity of eMZV relations. Nevertheless, the method of q-expansions allows to confirm that indeed, Fay and shuffle identities comprise the entirety of eMZV relations for a variety of combinations of weights and lengths. Accordingly, this leads us to conjecture that *all* available relations between eMZVs are implied by Fay and shuffle identities. At lengths and weights beyond the reach of our current computer implementation of Fay and shuffle identities, the comparison of q-expansions gives rise to conjectural relations which nicely tie in with the algebraic considerations to be described next.

In order to overcome the shortcomings of comparing q-expansions of eMZVs, one uses their differential equation in τ to write eMZVs as linear combinations of iterated integrals over Eisenstein series or *iterated Eisenstein integrals* for short. Contrary to the definition of eMZVs, where the iterated integration is performed over coordinates of the elliptic curve, the integration in their representation via Eisenstein series is performed over the modular parameter of the elliptic curve. Similar iterated integrals, some of them involving more general modular forms, have been studied by Manin in ref. [17]. Those integrals have been revisited by Brown [18] recently, who used them to define *multiple modular values*. A new feature of Brown's approach to iterated integrals of modular forms is that it allows also for an integration along paths which connect two cusps. Among other things, multiple modular values provide a conceptual explanation of the relationship between double zeta values and cusp forms [19].

The representation of eMZVs as iterated Eisenstein integrals is particularly convenient because the latter are believed to be linearly independent over the complex numbers. This has been tested up to the lengths and weights considered in this paper, but it remains a working hypothesis for several constructions in this work[1]. In particular, an analogue of the Fay relations is not known to hold for iterated Eisenstein integrals. This led to the first expectation that one can find the number of indecomposable eMZVs by enumerating all shuffle-independent iterated

[1]It is expected that $\mathbb{C}$-linear independence of iterated Eisenstein integrals holds in full generality, and it can presumably be shown using the techniques introduced in [20]. We do not attempt a proof in this paper and defer this problem to future work [21].

Eisenstein integrals.

While this idea indeed yields the correct number of indecomposable eMZVs of low length and weight, there is another effect appearing at higher weight: in the rewriting of *any* eMZV certain shuffle-independent iterated Eisenstein integrals occur in rigid linear combinations only. In other words, not all iterated Eisenstein integrals can be expressed in terms of eMZVs. The above rigid linear combinations in turn are implied by relations well known from a special algebra of derivations $\mathfrak{u}$ [22–25]. The patterns we find from investigating various eMZVs exactly match the available data about the derivation algebra in refs. [24, 26]. Consequently, we turn this into a method to infer the number of indecomposable eMZVs at given weight and length. The results of this method agree perfectly with the data obtained from either shuffle and Fay relations. In addition, complete knowledge of relations in the derivation algebra leads to upper bounds on the number of indecomposable eMZVs. Those upper bounds complement the lower bounds obtained from comparing q-expansions. The study of q-expansions allows to enumerate indecomposable eMZVs of given weight and length without assuming linear independence of different iterated Eisenstein integrals, which will be discussed below.

The algebra of derivations $\mathfrak{u}$ on one side and eMZVs on the other side are linked by a differential equation for the elliptic KZB associator [22, 23] derived by Enriquez [16]. This differential equation implies upper bounds on the number of indecomposable eMZVs. Assuming these upper bounds to be attained, one can extend the knowledge about the derivation algebra $\mathfrak{u}$ substantially: we identify numerous apparently new relations up to and including depth five. Moreover, there is no conceptual bottleneck in extending the analysis to arbitrary weight and depth.

The link between eMZVs and the derivation algebra $\mathfrak{u}$ becomes particularly clear upon mapping iterated Eisenstein integrals onto words composed from non-commutative letters g. The viability of the bookkeeping framework introduced in section 4 relies on the linear independence of iterated Eisenstein integrals, which is not proved at this point. On the other hand, since one can always check linear independence for a finite number of iterated Eisenstein integrals directly, the empirical results made available on our eMZV webpage do not depend on the general linear independence statement.

Moreover, the rewriting of eMZVs as linear combinations of iterated Eisenstein integrals and into letters g lateron bears similarities to a process, which appeared already in the context of usual MZVs. Namely, employing a conjectural isomorphism ϕ, MZVs can be rewritten in terms of an alphabet of non-commutative letters f (cf. ref. [11]). However, the construction of the map ϕ is highly elaborate, as it requires the motivic coaction and also depends on the choice of an algebra basis for (motivic) MZVs. On the contrary, the rewriting of eMZVs in terms of iterated Eisenstein integrals is completely canonical and straightforward from the differential equation for eMZVs. On the other hand, while the number of indecomposable MZVs at given weight is determined by counting *all* shuffle-independent words in f, the corresponding problem for eMZVs requires the consideration of the non-trivial relations in the derivation algebra $\mathfrak{u}$ in addition.

In summary, the results in this work are fourfold:

- an explicit basis of irreducible eMZVs of certain weights and lengths, together with expressions of eMZVs in that basis collected on a web page `https://tools.aei.mpg.de/emzv`

- the observation that Fay, shuffle and reflection relations generate all identities between eMZVs up to the weights and lengths considered.

- a general method for counting irreducible eMZVs based on iterated Eisenstein integrals and the special derivation algebra, for which we need to assume linear independence of iterated Eisenstein integrals.
- several explicit new relations in the special derivation algebra $\mathfrak{u}$, which are collected also at `https://tools.aei.mpg.de/emzv`

The article is organized as follows: section 2 starts with a small review of eMZVs and sets the stage for combining Fay and shuffle relations with the q-expansion, resulting in an "observational" set of indecomposable eMZVs. Section 3 is devoted to a brief recapitulation of structures present for usual MZVs with particular emphasis on their rewriting in terms of non-commutative words. In section 4 we set up the translation of eMZVs into iterated Eisenstein integrals, investigate their properties and connect the bookkeeping of indecomposable eMZVs with the algebra of derivations $\mathfrak{u}$. In subsection 4.6 we describe a modified version of iterated Eisenstein integrals suitable in particular for the description of eMZVs. Several appendices are complementary to the discussion in the main text. In particular, some relations between elements of the derivation algebra are collected in appendix C.2.

2 Relations between elliptic multiple zeta values

After recalling the definition of eMZVs, we are going to explore the implications of Fay and shuffle relations as well as the method of q-expansions. In addition, we will describe how usual MZVs defined via

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \ , \qquad n_r \geq 2 \tag{2.1}$$

arise as constant terms of eMZVs.

2.1 Prerequisites and definitions

In this subsection we will briefly review elliptic iterated integrals and define eMZVs. An elaborate introduction from a string theorist's point of view is available in ref. [5]. To get started, let us consider iterated integrals on the punctured elliptic curve $E_\tau^\times$, which is $E_\tau \equiv \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with the origin removed and $\mathrm{Im}\,(\tau) > 0$. We will frequently refer to the modular parameter τ by its exponentiated version

$$q \equiv e^{2\pi i\tau}, \quad \text{such that} \quad 2\pi i \frac{\mathrm{d}}{\mathrm{d}\tau} = -4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q} = -4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q} \ . \tag{2.2}$$

Functions A of the modular parameter will be denoted by either $A(\tau)$ or $A(q)$.

Weighting functions. A natural collection of weighting functions for the iterated integration to be defined below is provided by the Eisenstein-Kronecker series $F(z,\alpha,\tau)$ [27, 4],

$$F(z,\alpha,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)} \ , \tag{2.3}$$

where θ_1 is the odd Jacobi theta function and the tick denotes a derivative with respect to the first argument. The definition eq. (2.3) immediately yields $F(z,\alpha,\tau) = F(z+1,\alpha,\tau)$, and supplementing an additional, non-holomorphic factor lifts the quasi-periodicity of the Eisenstein-Kronecker series with respect to $z \mapsto z+\tau$ to an honest double-periodicity. The resulting function $\Omega(z,\alpha,\tau)$ on an elliptic curve serves as a generating series for the weighting functions $f^{(n)}(z,\tau)$ in eMZVs:

$$\Omega(z,\alpha,\tau) \equiv \exp\left(2\pi i\alpha \frac{\operatorname{Im}(z)}{\operatorname{Im}(\tau)}\right) F(z,\alpha,\tau) = \sum_{n=0}^{\infty} f^{(n)}(z,\tau)\alpha^{n-1} \, . \tag{2.4}$$

The functions $f^{(n)}$ are doubly periodic and alternate in their parity,

$$f^{(n)}(z+1,\tau) = f^{(n)}(z+\tau,\tau) = f^{(n)}(z,\tau) \, , \qquad f^{(n)}(-z,\tau) = (-1)^n f^{(n)}(z,\tau) \, . \tag{2.5}$$

Their simplest instances read

$$f^{(0)}(z,\tau) = 1 \, , \qquad f^{(1)}(z,\tau) = \frac{\theta_1'(z,\tau)}{\theta_1(z,\tau)} + 2\pi i \frac{\operatorname{Im}(z)}{\operatorname{Im}(\tau)} \, , \tag{2.6}$$

and $f^{(1)}$ is in fact the only weighting function with a simple pole on the lattice $\mathbb{Z}+\mathbb{Z}\tau$ including the origin. The remaining $f^{(n)}$ with $n \neq 1$ are non-singular on the entire elliptic curve. As elaborated in [4] and section 3 of [5], the weighting functions $f^{(n)}$ can be expressed in terms of Eisenstein functions and series the latter of which will play a central rôle in the sections below.

Elliptic iterated integrals and eMZVs. Even though the functions $f^{(n)}$ are defined for arbitrary complex arguments z and suitable for integrations along both homology cycles of the elliptic curve, we will restrict our subsequent discussion to real arguments. This is sufficient for studying eMZVs as iterated integrals over the interval $[0,1]$ on the real axis and avoids the necessity for homotopy-invariant completions of the integrands[2]. Hence, any integration variable and first argument of $f^{(n)}(z,\tau)$ is understood to be real.

Employing the functions $f^{(n)}$, iterated integrals on the elliptic curve $E_\tau^\times$ are defined via

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix}; z\right) \equiv \int_0^z \mathrm{d}t\, f^{(n_1)}(t-a_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix}; t\right) , \tag{2.7}$$

where the recursion starts with $\Gamma(;z) \equiv 1$. The elliptic iterated integral in eq. (2.7) is said to have *weight* $w = \sum_{i=1}^r n_i$, and the number r of integrations will be referred to as its *length*. Beginning with the above equation, we will usually suppress the second argument τ for the weighting functions $f^{(n)}$ and the elliptic iterated integrals Γ.

Definition (2.7) implies a shuffle relation with respect to the combined letters $A_i \equiv \begin{smallmatrix} n_i \\ a_i \end{smallmatrix}$ describing the weighting functions $f^{(n_i)}(z-a_i)$,

$$\Gamma(A_1,A_2,\dots,A_r;z)\,\Gamma(B_1,B_2,\dots,B_q;z) = \Gamma\left((A_1,A_2,\dots,A_r) \shuffle (B_1,B_2,\dots,B_q);z\right) . \tag{2.8}$$

Another obvious property of elliptic iterated integrals is the reflection identity due to eq. (2.5)

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix}; z\right) = (-1)^{n_1+n_2+\dots+n_r}\, \Gamma\left(\begin{smallmatrix} n_r & \dots & n_2 & n_1 \\ z-a_r & \dots & z-a_2 & z-a_1 \end{smallmatrix}; z\right) . \tag{2.9}$$

Finally, if all the labels a_i vanish – which, because of the periodicity of $f^{(n)}$ is equivalent to all

[2]A generating series for homotopy-invariant iterated integrals is given in ref. [4], in which the differential forms $f^{(n)}(z,\tau)\mathrm{d}z$ are accompanied by $\nu \equiv 2\pi i \frac{\mathrm{dIm}\,(z)}{\operatorname{Im}(\tau)}$. While any integral based upon a sequence of ν and $\mathrm{d}z$ has a unique homotopy-invariant uplift via admixtures of $f^{(n>0)}(z,\tau)\mathrm{d}z$, iterated integrals of $f^{(n)}(z,\tau)\mathrm{d}z$ allow for multiple homotopy-invariant completions via ν. A thorough discussion of the issue is provided in ref. [5].

labels a_i being integer – we will often use the notation

$$\Gamma(n_1, n_2, \ldots, n_r; z) \equiv \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ 0 & 0 & \ldots & 0 \end{smallmatrix} ; z \right) . \tag{2.10}$$

Evaluating the elliptic iterated integrals in eq. (2.10) at $z = 1$ gives rise to *elliptic multiple zeta values* or *eMZV*s for short [1]:

$$\begin{aligned} \omega(n_1, n_2, \ldots, n_r) &\equiv \int\limits_{0 \le z_i \le z_{i+1} \le 1} f^{(n_1)}(z_1) \mathrm{d}z_1 \, f^{(n_2)}(z_2) \mathrm{d}z_2 \, \ldots f^{(n_r)}(z_r) \mathrm{d}z_r \\ &= \Gamma(n_r, \ldots, n_2, n_1; 1) , \end{aligned} \tag{2.11}$$

where $\ell_\omega = r$ is referred to as the *length* while $w_\omega = \sum_{i=1}^{r} n_i$ is called the *weight* of an eMZV. The subscript ω refers to the current ω-representation of eMZVs in eq. (2.11), which has a different notion of weight and length compared to the iterated Eisenstein integrals to be defined below in section 4.

Being defined on an elliptic curve, eMZVs depend on its modular parameter τ and furnish a natural genus-one generalization of standard MZVs, which are to be reviewed in section 3.

The shuffle relation eq. (2.8) straightforwardly carries over to eMZVs,

$$\omega(n_1, n_2, \ldots, n_r) \, \omega(k_1, k_2, \ldots, k_s) = \omega\big((n_1, n_2, \ldots, n_r) \sqcup\!\sqcup (k_1, k_2, \ldots, k_s)\big) , \tag{2.12}$$

whereas the parity property eq. (2.5) of the weighting functions $f^{(n)}$ implies the reflection identity

$$\omega(n_1, n_2, \ldots, n_{r-1}, n_r) = (-1)^{n_1 + n_2 + \ldots + n_r} \, \omega(n_r, n_{r-1}, \ldots, n_2, n_1) . \tag{2.13}$$

Note that this implies the vanishing of odd-weight eMZVs with reversal-symmetric labels:

$$\omega(n_1, n_2, \ldots, n_r) = 0 , \qquad \text{if } (n_1, n_2, \ldots, n_r) = (n_r, \ldots, n_2, n_1) \text{ and } \sum_{i=1}^{r} n_i \text{ odd} . \tag{2.14}$$

Although suppressed in our notation, every eMZV is still a function of the modular parameter τ and inherits a Fourier expansion in q from the restriction of $f^{(n)}$ to real arguments.

$$\omega(n_1, \ldots, n_r) = \omega_0(n_1, \ldots, n_r) + \sum_{k=1}^{\infty} c_k(n_1, \ldots, n_r) q^k . \tag{2.15}$$

The τ-independent quantity ω_0 in eq. (2.15) is called the *constant term* of ω and will be shown to consist of MZVs and integer powers of $2\pi i$ in the next subsection. We will refer to eMZVs for which $c_k(n_1, \ldots, n_r) = 0$ for all $k \in \mathbb{N}^+$ as *constant.*

Regularization. While the functions $f^{(n)}(z)$ are smooth for $n \neq 1$, the function $f^{(1)}(z)$ in eq. (2.6) diverges as $\frac{1}{z}$ and $\frac{1}{z-1}$ for $z \to 0$ and $z \to 1$, respectively. Hence, eMZVs $\omega(n_1, \ldots, n_r)$ with $n_1 = 1$ or $n_r = 1$ are a priori divergent, and require a regularization process similar to shuffle regularization for MZVs [1] (cf. also [21]). A natural choice at genus one is to modify the integration region in eq. (2.7) by a small $\varepsilon > 0$,

$$\int\limits_{\varepsilon \le z_i \le z_{i+1} \le z - \varepsilon} f^{(n_1)}(z_1 - a_1) \mathrm{d}z_1 \, f^{(n_2)}(z_2 - a_2) \mathrm{d}z_2 \, \ldots f^{(n_r)}(z_r - a_r) \mathrm{d}z_r , \tag{2.16}$$

and to expand the integral as a polynomial in $\log(-2\pi i\varepsilon)$. Hereby the branch of the logarithm is chosen such that $\log(-i) = -\frac{\pi i}{2}$. The regularized value of eq. (2.16) is then defined to be the constant term in the ε-expansion. The factor $-2\pi i$ in the expansion parameter $\log(-2\pi i\varepsilon)$ ensures that the limit $\tau \to i\infty$ does not introduce any logarithms, and that eMZVs degenerate to MZVs upon setting $z = 1$ in eq. (2.16). For later reference, we will call eMZVs of the form $\omega(1, n_2, \ldots)$ or $\omega(\ldots, n_{r-1}, 1)$ *divergent*.

For the enumeration of eMZVs, we have employed an infinite alphabet, consisting of the non-negative integers $0, 1, 2, \ldots$ eq. (2.11). There is another way of carrying out this enumeration, which uses a two-letter alphabet instead [4]. The two-letter alphabet descends from a construction of eMZVs via *homotopy-invariant* iterated integrals. Since every eMZV in the infinite alphabet can be rewritten as an eMZV in the two-letter alphabet and vice-versa, one does not lose information by choosing to work with one alphabet or the other.

2.2 Fay and shuffle relations

In this subsection, we analyze relations among eMZVs defined in eq. (2.11) and gather examples of *indecomposable* eMZVs. A set of *indecomposable* eMZVs of weight w_ω and length ℓ_ω is a *minimal* set of eMZVs such that any other eMZV of the same weight and length can be expressed as a linear combination of elements from this set and

- products of eMZVs with strictly positive weights,
- eMZVs of lengths smaller than ℓ_ω or weight lower than w_ω,

where the coefficients comprise MZVs (including rational numbers) and integer powers of $2\pi i$. After exploring the consequences of shuffle and reflection identities eqs. (2.12) and (2.13), Fay identities are discussed as a genus-one analogue of the partial-fraction identities among products of $(z-a)^{-1}$, which arise from the differential forms $\mathrm{dlog}(z-a)$. The weight of eMZVs is preserved under all these identities whereas the length obviously varies in Fay and shuffle relations. In contradistinction to usual MZVs, the availability of $f^{(0)} \equiv 1$ as a weighting function yields an infinite number of eMZVs for a certain weight, so the counting of indecomposable eMZVs must be performed at fixed length and weight.

Examples of constant eMZVs. The simplest examples of the eMZVs defined in eq. (2.11) are of length one:

$$\omega(n_1) = \begin{cases} -2\,\zeta_{n_1} & : \ n_1 \text{ even} \\ 0 & : \ n_1 \text{ odd} \end{cases} . \tag{2.17}$$

The underlying single integration over the interval $[0, 1]$ picks up the constant term in the q-expansion of $f^{(n)}$ (see section 3.3 of ref. [5]) and yields the constants in eq. (2.17) with regularized value $\zeta_0 = -\frac{1}{2}$ in $\omega(0) = 1$.

Another distinction between even and odd labels n_i occurs at length $\ell_\omega = 2$. The union of shuffle and reflection identities eqs. (2.12) and (2.13) contains more independent relations for even total weight than for odd weight, and the eMZVs are then completely determined by eq. (2.17):

$$\omega(n_1, n_2)\Big|_{n_1+n_2 \text{ even}} = \begin{cases} 2\,\zeta_{n_1}\,\zeta_{n_2} & : \ n_1, n_2 \text{ even} \\ 0 & : \ n_1, n_2 \text{ odd} \end{cases} . \tag{2.18}$$

For eMZVs of odd total weight, on the other hand, shuffle and reflection relations at length two coincide, and $\omega(n_1, n_2)$ are no longer bound to be constant. This correlation between $(-1)^{w_\omega}$

and the length will be turned into a general rule in the next paragraph. In addition, there are also constant eMZVs, which make their appearance only at sufficiently high length. For example, ζ_3 is identified in eq. (2.36) to be an eMZV of weight 3 and length 4. One can show that all constant eMZVs evaluate to products of MZVs and integer powers of $2\pi i$, see Proposition 5.3 of ref. [1].

Interesting and boring eMZVs. The lack of a τ-dependence for eMZVs $\omega(n_1, n_2)$ of even weight can be viewed as the analogue of the vanishing of $\omega(n_1)$ for odd weight as observed in eq. (2.17). The general pattern is as follows: whenever weight and length of an eMZV have the same parity (i.e. $(-1)^{w_\omega} = (-1)^{\ell_\omega}$), shuffle and reflection identities eqs. (2.12) and (2.13) allow to determine this eMZV in terms of eMZVs of lower length. Novel indecomposable eMZVs can only occur for opposite parity $(-1)^{w_\omega} = -(-1)^{\ell_\omega}$ such as the odd-weight $\omega(n_1, n_2)$. Accordingly, an eMZV $\omega(n_1, n_2, \ldots, n_r)$ is called *interesting*, if the combination $w_\omega + \ell_\omega$ of weight and length is odd, otherwise we refer to it as *boring*.

Boring eMZVs at length $\ell_\omega = 3$ can arise from four different choices of even and odd labels. For those, the shuffle identities eq. (2.12) allow to reduce them to interesting eMZVs at length two. Explicitly, we have

$$\begin{aligned}
\omega(o_1, o_2, o_3) &= 0 \\
\omega(e_1, e_2, o_3) &= -\,\zeta_{e_1}\,\omega(e_2, o_3) \\
\omega(e_1, o_2, e_3) &= -\,\zeta_{e_1}\,\omega(o_2, e_3) - \zeta_{e_3}\,\omega(e_1, o_2) \\
\omega(o_1, e_2, e_3) &= -\,\zeta_{e_3}\,\omega(o_1, e_2)\,,
\end{aligned} \tag{2.19}$$

where e_i and o_i refer to even and odd labels, respectively. Similarly, boring eMZVs at length $\ell_\omega = 4$ come in the following (reflection-independent) classes:

$$\begin{aligned}
\omega(o_1, o_2, o_3, o_4) &= 0 \\
\omega(e_1, e_2, e_3, e_4) &= -2\,\zeta_{e_1}\,\zeta_{e_2}\,\zeta_{e_3}\,\zeta_{e_4} - \zeta_{e_4}\,\omega(e_1, e_2, e_3) - \zeta_{e_1}\,\omega(e_2, e_3, e_4) \\
\omega(o_1, o_2, e_3, e_4) &= -\,\zeta_{e_4}\,\omega(o_1, o_2, e_3) \\
\omega(o_1, e_2, o_3, e_4) &= \tfrac{1}{2}\,\omega(o_1, e_2)\,\omega(o_3, e_4) - \zeta_{e_4}\,\omega(o_1, e_2, o_3) \\
\omega(o_1, e_2, e_3, o_4) &= \tfrac{1}{2}\,\omega(o_1, e_2)\,\omega(e_3, o_4) \\
\omega(e_1, o_2, o_3, e_4) &= \tfrac{1}{2}\,\omega(e_1, o_2)\,\omega(o_3, e_4) - \zeta_{e_1}\,\omega(o_2, o_3, e_4) - \zeta_{e_4}\,\omega(e_1, o_2, o_3)\,.
\end{aligned} \tag{2.20}$$

Although becoming more involved for higher length, the distinction of cases as well as the decomposition of boring eMZVs can be cast into a nice form, as is explained in appendix A.1. Below, however, we will be concerned with interesting eMZVs mostly. Note that the vanishing of eMZVs with only odd entries is true at all lengths,

$$\omega(o_1, o_2, \ldots, o_r) = 0\ . \tag{2.21}$$

Fay relations among $f^{(n)}$ and elliptic iterated integrals. While reflection and shuffle identities preserve the partition of the modular weight among the integrated $f^{(n_i)}$, so-called Fay relations mix eMZVs involving different values of n_i. They can be traced back to the Fay identity of their generating series eq. (2.4) [4]

$$\Omega(z_1, \alpha_1, \tau)\Omega(z_2, \alpha_2, \tau) = \Omega(z_1, \alpha_1 + \alpha_2, \tau)\Omega(z_2 - z_1, \alpha_2, \tau)$$

$$+\,\Omega(z_2,\alpha_1+\alpha_2,\tau)\Omega(z_1-z_2,\alpha_1,\tau)\ , \tag{2.22}$$

which is valid for any complex z_1, z_2 and follows from the Fay trisecant equation [28]. Relations among $f^{(n)}$ can be read off from eq. (2.22) by isolating monomials in α_1, α_2 [5]

$$f^{(n_1)}(t-x)f^{(n_2)}(t) = -(-1)^{n_1} f^{(n_1+n_2)}(x) + \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} f^{(n_2-j)}(x) f^{(n_1+j)}(t-x)$$
$$+ \sum_{j=0}^{n_1} \binom{n_2-1+j}{j} (-1)^{n_1+j} f^{(n_1-j)}(x) f^{(n_2+j)}(t)\ . \tag{2.23}$$

The simplest instance of these Fay relations can be viewed as a genus-one counterpart of partial-fraction relations such as $\frac{1}{tx} = \frac{1}{x(t-x)} + \frac{1}{t(x-t)}$:

$$f^{(1)}(t-x)f^{(1)}(t) = f^{(1)}(t-x)f^{(1)}(x) - f^{(1)}(t)f^{(1)}(x) + f^{(2)}(t) + f^{(2)}(x) + f^{(2)}(t-x)\ . \tag{2.24}$$

The Fay relations eq. (2.23) are a very powerful tool for rearranging the elliptic iterated integrals in eq. (2.7). Together with the derivatives of Γ with respect to their argument z and labels a_i [5], they allow for example to recursively remove any appearance of $a_i = z$ in the label of an iterated integral, e.g.

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ z & 0 & \dots & 0 \end{smallmatrix}; z\right) = (-1)^r \zeta_r \prod_{j=1}^{r} \delta_{n_j,1} - (-1)^{n_1} \Gamma\left(\begin{smallmatrix} n_1+n_2 & 0 & n_3 & \dots & n_r \\ 0 & 0 & 0 & \dots & 0 \end{smallmatrix}; z\right)$$
$$+ \sum_{j=0}^{n_1} (-1)^{n_1+j} \binom{n_2-1+j}{j} \Gamma\left(\begin{smallmatrix} n_1-j & n_2+j & n_3 & \dots & n_r \\ 0 & 0 & 0 & \dots & 0 \end{smallmatrix}; z\right)$$
$$+ \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} \int_0^z \mathrm{d}t\, f^{(n_2-j)}(t)\, \Gamma\left(\begin{smallmatrix} n_1+j & n_3 & \dots & n_r \\ t & 0 & \dots & 0 \end{smallmatrix}; t\right)\ , \tag{2.25}$$

see appendix A.2 for a generalization to multiple appearances of $a_i = z$. The zeta value ζ_r in the first line of eq. (2.25) stems from the limit $z \to 0$ of the left hand side for which $f^{(1)}(z)$ can be approximated by $\frac{1}{z}$ [5]. Note that the Kronecker-deltas $\delta_{n_j,1}$ ensure that the notions of weights for MZVs and elliptic iterated integrals are compatible in eq. (2.25).

Fay relations among eMZVs. A rich class of eMZV relations can be inferred from the limit $z \to 1$ of eq. (2.25). On the left hand side, periodicity of $f^{(n)}$ w.r.t. $z \to z+1$ leads to

$$\lim_{z\to 1} \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ z & 0 & \dots & 0 \end{smallmatrix}; z\right) = \omega(n_r, \dots, n_2, n_1)\ , \qquad n_1 \neq 1 \text{ or } n_2 \neq 1\ , \tag{2.26}$$

where cases with $n_1 = n_2 = 1$ require an additional treatment of the poles of the associated $f^{(1)}$ and are therefore excluded[3]. By eq. (2.11), the elliptic iterated integrals on the right hand side reduce to eMZVs under $z \to 1$ once the recursion eq. (2.25) has been applied iteratively to remove any appearance of the argument from the labels. At length two, the resulting eMZV relation is

$$\omega(n_2, n_1) = -(-1)^{n_1} \omega(0, n_1+n_2) + \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} (-1)^{n_1+j} \omega(n_1+j, n_2-j)$$

[3]For cases with $n_1 = 1$ and $n_2 \neq 1$, we could not prove the general absence of extra contributions from the poles of $f^{(1)}$. However, the validity of eq. (2.26) in these cases has been thoroughly tested to lengths $r \leq 6$ using the methods in section 2.3. Hence, eq. (2.26) at $n_1 = 1$ and $n_2 \neq 1$ with general r remains a well-tested conjecture.

$$+\sum_{j=0}^{n_1}\binom{n_2-1+j}{j}(-1)^{n_1+j}\,\omega(n_2+j,n_1-j)\ , \qquad n_1\neq 1 \text{ or } n_2\neq 1\ , \quad (2.27)$$

and length three requires two applications of the recursion in eq. (2.25):

$$\begin{aligned}\omega(n_3,n_2,n_1) &= \zeta_2\sum_{j=0}^{n_2}\delta_{n_3,1}\delta_{n_1+j,1}\binom{n_1-1+j}{j}\,\omega(n_2-j)\\ &-(-1)^{n_1}\,\omega(n_3,0,n_1+n_2)+\sum_{j=0}^{n_1}(-1)^{n_1+j}\binom{n_2-1+j}{j}\,\omega(n_3,n_2+j,n_1-j)\\ &+\sum_{j=0}^{n_2}\binom{n_1-1+j}{j}\sum_{k=0}^{n_3}(-1)^{n_1+j+k}\binom{n_1+j-1+k}{k}\,\omega(n_1+j+k,n_3-k,n_2-j)\\ &+\sum_{j=0}^{n_2}\binom{n_1-1+j}{j}\sum_{k=0}^{n_1+j}(-1)^{n_1+j+k}\binom{n_3-1+k}{k}\,\omega(n_3+k,n_1+j-k,n_2-j)\\ &-\sum_{j=0}^{n_2}(-1)^{n_1+j}\binom{n_1-1+j}{j}\,\omega(0,n_1+n_3+j,n_2-j)\ , \qquad n_1\neq 1 \text{ or } n_2\neq 1\ .\end{aligned} \quad (2.28)$$

It is straightforward to derive higher-length relations (involving any ζ_r with $2\leq r\leq \ell_\omega-1$) from further iterations of eq. (2.25) in the limit $z\to 1$. The exclusion of $n_1=n_2=1$ suppresses ζ_2 in eq. (2.27) and ζ_3 in eq. (2.28), and, more generally, the appearance of ζ_r is relegated to eMZV relations of length $r+1$. By the relations derived from $\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_k & n_{k+1} & \ldots & n_r \\ z & z & \ldots & z & 0 & \ldots & 0\end{smallmatrix};z\right)$ in appendix A.2, analogous statements apply to generic MZVs, and any MZV will appear in the rewriting of some $\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r\end{smallmatrix};z\right)$ with appropriate combinations of $a_j\in\{0,z\}$.

Combining shuffle and Fay relations. Shuffle relations reduce boring eMZVs to interesting eMZVs of lower length, see for instance eqs. (2.18) and (2.19). At first glance, this appears to attribute more significance to Fay relations among interesting eMZVs, e.g. eq. (2.27) at odd n_1+n_2 and eq. (2.28) at even $n_1+n_2+n_3$. The former yields length-two relations such as

$$\omega(0,5)=\omega(2,3)\ , \qquad \omega(3,4)=-2\,\omega(0,7)+\omega(2,5)\ , \quad (2.29)$$

which by themselves leave $1+\lfloor\frac{1}{3}(n_1+n_2)\rfloor$ eMZVs at length $\ell_\omega=2$ and weight $w_\omega=n_1+n_2$ independent [29]. However, Fay relations eq. (2.28) among boring eMZVs at length three turn out to contain additional information about interesting $\omega(n_1,n_2)$. For example, writing eq. (2.28) with $(n_1,n_2,n_3)=(1,0,2)$,

$$\omega(0,3,0)-\omega(1,2,0)-\omega(2,0,1)+\omega(3,0,0)=0\ , \quad (2.30)$$

followed by a shuffle-reduction of the boring eMZVs via eq. (2.19) yields the length-two relation

$$\omega(1,2)=2\,\zeta_2\,\omega(0,1)-\omega(0,3)\ . \quad (2.31)$$

This relation would be inaccessible from Fay relations at length two and identifies $\omega(0,3)$ to be the unique indecomposable $\omega(n_1,n_2)$ at weight three, which is short of the above $1+\lfloor\frac{1}{3}(n_1+n_2)\rfloor$ counting. Hence – when combined with shuffle-relations – Fay relations among boring eMZVs at length $\ell_\omega+1$ provide more information than their counterparts among interesting eMZVs at length ℓ_ω. The need for Fay relations at length $\ell_\omega+1$ to classify indecomposable eMZVs at length ℓ_ω is reminiscent of double-shuffle relations among MZVs. For example, the relation

$$\zeta_{5,7} = \frac{14}{9}\,\zeta_{3,9} + \frac{28}{3}\,\zeta_5\,\zeta_7 - \frac{121285}{12438}\,\zeta_{12} \tag{2.32}$$

is inaccessible from double-shuffle relations of depth two and requires higher-depth input [19].

Indecomposable eMZVs. By applying the shuffle-reduction eq. (2.19) to higher-weight instances of the length-three Fay relation eq. (2.28), any length-two eMZV can be expressed in terms of products of ζ_{2k} and $\omega(0, 2n-1)$:

$$\begin{aligned} \omega(n_1,n_2)\Big|_{n_1+n_2\ \text{odd}} &= (-1)^{n_1}\,\omega(0,n_1+n_2) + 2\delta_{n_1,1}\,\zeta_{n_2}\,\omega(0,1) - 2\delta_{n_2,1}\,\zeta_{n_1}\,\omega(0,1) \\ &\quad + 2\sum_{p=1}^{\lceil\frac{1}{2}(n_2-3)\rceil} \binom{n_1+n_2-2p-2}{n_1-1}\,\zeta_{n_1+n_2-2p-1}\,\omega(0,2p+1) \\ &\quad - 2\sum_{p=1}^{\lceil\frac{1}{2}(n_1-3)\rceil} \binom{n_1+n_2-2p-2}{n_2-1}\,\zeta_{n_1+n_2-2p-1}\,\omega(0,2p+1)\,, \end{aligned} \tag{2.33}$$

which implies that no eMZVs at length two other than $\omega(0, 2n-1)$ are indecomposable. This relation can be straightforwardly proven using the techniques of subsection 2.3.

Accordingly, the richest source of relations between interesting eMZVs at length three are the length-four Fay relations at even weight together with the shuffle reduction eq. (2.20) of the boring eMZVs therein. The indecomposable eMZVs can be chosen to include $\omega(0, 0, 2n)$ by analogy with eq. (2.33), and additional indecomposable eMZVs such as $\omega(0, 3, 5)$ occur at weight $w_\omega \geq 8$, e.g.

$$\begin{aligned} \omega(1,1,2) &= \frac{13}{12}\,\zeta_4 - \zeta_2\,\omega(0,1)^2 + \omega(0,1)\,\omega(0,3) + 3\,\zeta_2\,\omega(0,0,2) - \frac{1}{2}\,\omega(0,0,4) \\ \omega(0,6,2) &= -\frac{21}{2}\,\zeta_8 + 2\,\omega(0,3)\,\omega(0,5) - 14\,\zeta_6\,\omega(0,0,2) - 6\,\zeta_4\,\omega(0,0,4) - \frac{9}{2}\,\omega(0,0,8) - \frac{2}{5}\,\omega(0,3,5)\,. \end{aligned} \tag{2.34}$$

Similarly, the set of indecomposable length-three eMZVs at weights ten and twelve can be chosen as $\{\omega(0, 0, 10), \omega(0, 3, 7)\}$ and $\{\omega(0, 0, 12), \omega(0, 3, 9)\}$, respectively. The weight-twelve relation

$$\begin{aligned} \omega(0,5,7) = &-140\,\zeta_{10}\,\omega(0,0,2) - 14\,\zeta_8\,\omega(0,0,4) + \frac{28}{3}\,\omega(0,5)\,\omega(0,7) \\ &- \frac{119}{6}\,\omega(0,0,12) + \frac{14}{9}\,\omega(0,3,9) - \frac{550396}{6219}\,\zeta_{12} \end{aligned} \tag{2.35}$$

will play an essential rôle later on.

While even-weight single MZVs are special cases of length-one eMZVs by eq. (2.17), odd MZVs do not show up in any relation for an eMZV of $\ell_\omega \leq 3$. When applying the above procedure to higher lengths, ζ_3 is identified to be an eMZV by length-four relations such as

$$\omega(0,1,2,0) = \frac{1}{4}\,\omega(0,3) - \frac{5}{2}\omega(0,0,0,3) - \frac{\zeta_3}{4}\,. \tag{2.36}$$

The appearance of ζ_3 in eMZV relations at length $\ell_\omega = 4, 5$ is governed by eq. (2.26) at $r = 4, 5$, and similar relations are expected to hold for any odd single zeta value by eq. (2.25) and eq. (A.10). Further support stems from the description of the eMZVs' constant terms through the Drinfeld associator [30–32] in eq. (2.43) below.

Usual MZVs show up in many relations between eMZVs such as eq. (2.36). While crucial for matching the constant term for the eMZVs in question, we will not count them as indecomposable eMZVs. Instead, they will arise as suitably chosen boundary conditions for a differential equation to be elaborated upon below.

Table 1 shows a possible (non-canonical) choice of indecomposable eMZVs for weights up to 14 and length up to and including five. The need for higher-length Fay relations increases the computational complexity in the classification of indecomposable eMZVs using the above procedure. Hence, comparing the τ-dependence will enter as an additional method in the next subsection to extend the results in the table to higher lengths and weights. Still, shuffle, reflection and Fay relations were assembled completely at $\ell_\omega = 2$, at $\ell_\omega = 3$ with $w_\omega \leq 14$, at $\ell_\omega = 4$ with $w_\omega \leq 9$ as well as at $\ell_\omega = 5$ with $w_\omega \leq 6$, and additional eMZV relations at those weights and lengths have been ruled out on the basis of their q-expansion. Continuing the search for

w_ω \ ℓ_ω	2	3	4
1	$\omega(0,1)$		$\omega(0,0,1,0)$
3	$\omega(0,3)$		$\omega(0,0,0,3)$
5	$\omega(0,5)$		$\omega(0,0,0,5)$
			$\omega(0,0,2,3)$
7	$\omega(0,7)$		$\omega(0,0,0,7)$
			$\omega(0,0,2,5)$
			$\omega(0,0,4,3)$
9	$\omega(0,9)$		$\omega(0,0,0,9)$
			$\omega(0,0,2,7)$
			$\omega(0,0,4,5)$
			$\omega(0,1,3,5)$
11	$\omega(0,11)$		$\omega(0,0,0,11)$
			$\omega(0,0,2,9)$
			$\omega(0,0,4,7)$
			$\omega(0,1,3,7)$
			$\omega(0,3,3,5)$
13	$\omega(0,13)$		$\omega(0,0,0,13)$
			$\omega(0,0,2,11)$
			$\omega(0,0,4,9)$
			$\omega(0,1,3,9)$
			$\omega(0,1,5,7)$
			$\omega(0,3,3,7)$
			$\omega(0,3,5,5)$

w_ω \ ℓ_ω	2	3	4	5
2		$\omega(0,0,2)$		$\omega(0,0,0,0,2)$
4		$\omega(0,0,4)$		$\omega(0,0,0,0,4)$
				$\omega(0,0,0,1,3)$
6		$\omega(0,0,6)$		$\omega(0,0,0,0,6)$
				$\omega(0,0,0,1,5)$
				$\omega(0,0,0,2,4)$
				$\omega(0,0,2,2,2)$
8		$\omega(0,0,8)$		$\omega(0,0,0,0,8)$
		$\omega(0,3,5)$		$\omega(0,0,0,1,7)$
				$\omega(0,0,0,2,6)$
				$\omega(0,0,1,2,5)$
				$\omega(0,0,2,2,4)$
10		$\omega(0,0,10)$		$\omega(0,0,0,0,10)$
		$\omega(0,3,7)$		$\omega(0,0,0,1,9)$
				and 7 more
12		$\omega(0,0,12)$		$\omega(0,0,0,0,12)$
		$\omega(0,3,9)$		$\omega(0,0,0,1,11)$
				$\omega(0,0,0,2,12)$
				and 11 more
14		$\omega(0,0,14)$		$\omega(0,0,0,0,14)$
		$\omega(0,3,11)$		$\omega(0,0,0,1,13)$
		$\omega(0,5,9)$		$\omega(0,0,0,2,12)$
				and many more

Table 1: A possible choice of indecomposable eMZVs up to weight 14 and length 5. A table containing the elements missing here is available at `https://tools.aei.mpg.de/emzv`.

indecomposable eMZVs as described in previous and subsequent subsections leads to table 2, in which the number of indecomposable eMZVs for a certain length and weight are noted. Basis rules for rewriting each eMZV in terms of those indecomposable elements can be obtained in digital form from the web page `https://tools.aei.mpg.de/emzv` and are available up to and including weights $30, 18, 12, 10$ for lengths $3, 4, 5, 6$, respectively.

ℓ_ω \ w_ω	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3	
4	1		1		2		3		4		5		7		8		10		x
5		1		2		4		6		9		13		x		x		x	
6	1		2		4		8		13		x		x		x		x		x
7		1		2		x		x		x		x		x		x		x	

Table 2: Number $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs at length ℓ_ω and weight w_ω.

2.3 Constant term and q-expansion

The system of relations discussed in the previous section did not require any information on the eMZVs' functional dependence on the modular parameter τ. In this section, we determine the Fourier expansion in $q = e^{2\pi i\tau}$ based on a first-order differential equation in τ along with a boundary value at $\tau \to i\infty$. This will not only provide crosschecks for the above eMZV-relations but will also lead to the more efficient approach to classifying indecomposable eMZVs at higher length and weight in section 4.

Constant term. The constant term of an eMZV can be determined explicitly using results of ref. [1]. By construction, the elliptic KZB associator $A(\tau)$ is the generating series of eMZVs,

$$e^{\pi i[y,x]}A(\tau) \equiv \sum_{r\geq 0}(-1)^r \sum_{n_1,n_2,\ldots,n_r\geq 0} \omega(n_1,n_2,\ldots,n_r)\mathrm{ad}_x^{n_r}(y)\ldots\mathrm{ad}_x^{n_2}(y)\mathrm{ad}_x^{n_1}(y) \ , \tag{2.37}$$

and captures the monodromy of the elliptic KZB equation [22, 23] along the path $[0,1]$. The prefactor $e^{\pi i[y,x]}$ is adjusted to the regularization scheme in eq. (2.16). The variables x and y generate a complete, free algebra $\mathbb{C}\langle\langle x,y\rangle\rangle$ of formal power series with complex coefficients, whose multiplication is the concatenation product, and the convention for the adjoint action is

$$\mathrm{ad}_x(y) \equiv [x,y] \ , \qquad \mathrm{ad}_x^n(y) = \underbrace{[x,\ldots[x,[x,}_{n\ \text{times}}y]]\ldots] \ . \tag{2.38}$$

Note that the appearance of eMZVs in eq. (2.37) along with non-commutative words in x and y allows for an alternative enumeration scheme using a two-letter alphabet, see subsection 2.1.

Enriquez proved that $A(\tau)$ admits the asymptotic expansion as $\tau \to i\infty$ [1]

$$A(\tau) = \Phi(\tilde{y},t)\, e^{2\pi i\tilde{y}}\, \Phi(\tilde{y},t)^{-1} + \mathcal{O}(e^{2\pi i\tau}) \ , \tag{2.39}$$

where $\mathcal{O}(e^{2\pi i\tau})$ refers to the non-constant terms in eq. (2.15) exclusively. In the above equation, the genus-one alphabet consisting of x, y is translated into a genus-zero alphabet involving

$$t \equiv [y,x] \ , \qquad \tilde{y} \equiv -\frac{\mathrm{ad}_x}{e^{2\pi i \mathrm{ad}_x}-1}(y) \ , \tag{2.40}$$

and Φ denotes the Drinfeld associator [30–32]

$$\Phi(e_0,e_1) \equiv \sum_{\hat{W}\in\langle e_0,e_1\rangle} \zeta^{ш}(W)\cdot \hat{W} \ . \tag{2.41}$$

The sum over $\hat{W} \in \langle e_0, e_1 \rangle$ includes all non-commutative words in letters e_0 and e_1, and the word W is obtained from $\hat{W}$ by replacing letters e_0 and e_1 by 0 and 1, respectively. Then, $\zeta^{ш}(W)$ denote shuffle-regularized MZVs [33] which are uniquely determined from eq. (3.1), the shuffle product eq. (3.2) and the definition $\zeta^{ш}(0) = \zeta^{ш}(1) = 0$ for words of length one. Consequently, the first few terms of $\Phi(e_0, e_1)$ are given by

$$\Phi(e_0, e_1) = 1 - \zeta_2 [e_0, e_1] - \zeta_3 [e_0 + e_1, [e_0, e_1]] + \dots \ . \tag{2.42}$$

From eqs. (2.37) and (2.39), the generating series of constant terms $\omega_0(n_1, ..., n_r)$ of eMZVs is immediately obtained as

$$\sum_{r \geq 0} (-1)^r \sum_{n_1, \dots, n_r \geq 0} \omega_0(n_1, \dots, n_r) \mathrm{ad}_x^{n_r}(y) \dots \mathrm{ad}_x^{n_1}(y) = e^{\pi i [y,x]} \, \Phi(\tilde{y}, t) \, e^{2\pi i \tilde{y}} \, \Phi(\tilde{y}, t)^{-1} \ . \tag{2.43}$$

In order to transfer information from the right hand side of eq. (2.43) to the constant terms of eMZVs on the left hand side, it remains to expand words in the alphabet $\{\tilde{y}, t\}$ in eq. (2.40) as formal series of words in the alphabet $\{\mathrm{ad}_x^n(y) \,|\, n \geq 0\}$ and then to compare the coefficients of both sides.

Perhaps surprisingly, the case where all $n_i \neq 1$ is very simple to treat. In that case, only the middle term $e^{2\pi i \tilde{y}}$ from eq. (2.43) yields a non-trivial contribution, and therefore we have

$$\omega_0(n_1, n_2, \dots, n_r)\Big|_{n_i \neq 1} = \begin{cases} 0 & \text{if at least one } n_i \text{ is odd, and all } n_i \neq 1 \\ \dfrac{1}{r!} \displaystyle\prod_{i=1}^{r} (-2\,\zeta_{n_i}) & \text{if all } n_i \text{ are even} \end{cases} \ . \tag{2.44}$$

In particular, one finds

$$\omega(\underbrace{0, 0, \dots, 0}_{n \text{ times}}) = \frac{1}{n!} \ , \tag{2.45}$$

which is perfectly in line with $f^{(0)} \equiv 1$. On the other hand, in presence of $n_i = 1$ at some places, a general formula for the constant term is very cumbersome. Simple instances include

$$\begin{gathered} \omega_0(1,0) = -\frac{i\pi}{2} \ , \qquad \omega_0(1,0,0) = -\frac{i\pi}{4} \ , \qquad \omega_0(1,0,0,0) = -\frac{i\pi}{12} - \frac{\zeta_3}{24\,\zeta_2} \\ \omega_0(0,1,1,0,0) = \frac{\zeta_2}{15} \ , \qquad \omega_0(1,0,1,1,0,0) = -\frac{i\pi\,\zeta_2}{30} - \frac{\zeta_3}{8} - \frac{17\,\zeta_5}{96\,\zeta_2} \end{gathered} \tag{2.46}$$

with generalizations in eq. (B.4). Replacing $n_i = 0$ in the above identities by even values $n_i = 2k$ amounts to multiplication with $-2\,\zeta_{2k}$ on the right hand side.

***q*-expansion.** The q-dependent terms in the expansion can be determined using the known form of the τ-derivative of eMZVs. In Théorème 3.3 of ref. [1], the derivative of a generating functional for eMZVs is presented, which translates as follows into derivatives of individual eMZVs in our conventions:

$$\begin{aligned} 2\pi i \frac{\mathrm{d}}{\mathrm{d}\tau} \, \omega(n_1, \dots, n_r) &= -4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q} \, \omega(n_1, \dots, n_r) \\ &= n_1 \, \mathrm{G}_{n_1+1} \, \omega(n_2, \dots, n_r) - n_r \, \mathrm{G}_{n_r+1} \, \omega(n_1, \dots, n_{r-1}) \\ &+ \sum_{i=2}^{r} \Big\{ (-1)^{n_i} (n_{i-1} + n_i) \, \mathrm{G}_{n_{i-1}+n_i+1} \, \omega(n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_r) \end{aligned} \tag{2.47}$$

$$
\begin{aligned}
&- \sum_{k=0}^{n_{i-1}+1} (n_{i-1}-k) \binom{n_i+k-1}{k} \mathrm{G}_{n_{i-1}-k+1}\, \omega(n_1,\dots,n_{i-2},k+n_i,n_{i+1},\dots,n_r) \\
&+ \sum_{k=0}^{n_i+1} (n_i-k) \binom{n_{i-1}+k-1}{k} \mathrm{G}_{n_i-k+1}\, \omega(n_1,\dots,n_{i-2},k+n_{i-1},n_{i+1},\dots,n_r) \Big\} \,.
\end{aligned}
$$

The *Eisenstein series* $\mathrm{G}_k \equiv \mathrm{G}_k(\tau)$ on the right hand side are defined by[4]

$$
\begin{aligned}
\mathrm{G}_0(\tau) &\equiv -1 \\
\mathrm{G}_k(\tau) &\equiv \begin{cases} \displaystyle\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^k} & : \ k>0 \text{ even}\,, \\ 0 & : \ k>0 \text{ odd}\,. \end{cases}
\end{aligned}
\tag{2.48}
$$

Positive even values of k admit a series expansion in the modular parameter:

$$
\mathrm{G}_k(\tau) = 2\,\zeta_k + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} \qquad k>0 \text{ even}\,. \tag{2.49}
$$

Using the above formulæ and the known expansion of the Eisenstein series G_k in eq. (2.49), one can recursively obtain the explicit q-expansion for any eMZV: The length of eMZVs on the right hand side of eq. (2.47) is decreased by one compared to the left hand side, and the recursion terminates with the constant eMZVs at length one given by eq. (2.17).

In addition, one finds from eq. (2.47) that only divergent eMZVs with $n_1 = 1$ or $n_r = 1$ lead to the non-modular G_2, see the discussion around eq. (2.16). In all other situations which lead to the non-modular G_2 in the last three lines, the respective terms cancel out. Not surprisingly, the interesting and boring character of eMZVs is preserved by eq. (2.47): the decreased length on the right hand side is compensated by an increased weight.

Also, note that the differential equation eq. (2.47) contains no MZV terms. In fact, the only way through which MZVs enter the stage of eMZVs is by means of the constant term eq. (2.39) of the KZB associator. As mentioned earlier, this constant term can be thought of as a boundary-value prescription for the differential equation eq. (2.47), thereby determining eMZVs uniquely.

eMZV relations from the q-expansion. Based on the q-expansions described above, relations between eMZVs can be checked and ruled out by comparing their Fourier representations. In practice, one writes down an ansatz comprised from interesting eMZVs and products thereof with uniform weight and an upper bound on the length of interest, each term supplemented with fudge coefficients.

Naïvely, one could calculate the q-expansions of all constituents up to a certain order $q^{N_{\max}}$ and impose a matching along with each Fourier mode q^n for $0 \le n \le N_{\max}$. This allows to fix the above fudge coefficients and to check the relations' validity up to – in principle – arbitrary order. In an early stage of the project, our computer implementation of this approach with $N_{\max} = 160$ was far more efficient compared to the analysis of reflection, shuffle and Fay identities and lead

[4]The case $k = 2$ requires the Eisenstein summation prescription

$$
\sum_{m,n\in\mathbb{Z}} a_{m,n} \equiv \lim_{N\to\infty} \lim_{M\to\infty} \sum_{n=-N}^{N} \sum_{m=-M}^{M} a_{m,n}\,.
$$

to substantial parts of the data shown in tables 1 and 2.

However, since the comparison of q-expansions has to be cut off at some chosen power of q, a proof for relations using the method is impossible by construction. Even worse, this naïve method fails to capture the structural insight from eq. (2.47) that any q-dependence in eMZVs stems from iterated integrals of Eisenstein series. This crucial property is exploited in section 4, confirming the entries of table 2 in a rigorous and conceptually by far more elegant manner.

While the naïve comparison of Fourier coefficients merely provides a lower bound for the number of indecomposable eMZVs for a given weight and length, the description of eMZVs in terms of iterated Eisenstein integrals in section 4 yields complementary upper bounds. Under the additional assumption that different iterated Eisenstein integrals are linearly independent, these upper bounds are indeed saturated. However, since we do not attempt to prove their linear independence, the naïve matching of Fourier coefficients closes the associated loophole at the weights and lengths under consideration.

3 Multiple zeta values and the ϕ-map

In this section we gather information on the structure of MZVs, which are to be compared with those found for eMZVs in section 4 below. While represented as nested sums in eq. (2.1) in section 2, they can alternatively be defined as iterated integrals

$$\begin{aligned} \zeta_{n_1,n_2,\ldots,n_r} &= \int\limits_{0\leq z_i\leq z_{i+1}\leq 1} \omega_1 \underbrace{\omega_0\ldots\omega_0}_{n_1-1}\omega_1\underbrace{\omega_0\ldots\omega_0}_{n_2-1}\ldots\omega_1\underbrace{\omega_0\ldots\omega_0}_{n_r-1} \\ &= \zeta(1\underbrace{0\ldots0}_{n_1-1}1\underbrace{0\ldots0}_{n_2-1}\ldots1\underbrace{0\ldots0}_{n_r-1}) \end{aligned} \tag{3.1}$$

over the differential forms $\omega_0 \equiv \frac{\mathrm{d}z}{z}$ and $\omega_1 \equiv \frac{\mathrm{d}z}{1-z}$ with all z_i on the real line. The MZV $\zeta_{n_1,\ldots,n_r}$ is said to have *weight* $w = \sum_{i=1}^r n_i$ and *depth* r. Written in terms of words W composed from the letters 0 and 1, which correspond to the differential forms ω_0 and ω_1 in eq. (3.1), respectively, ζ's satisfy the shuffle product:

$$\zeta(W_1)\,\zeta(W_2) = \zeta(W_1 ⧢ W_2)\,. \tag{3.2}$$

There is also a second product structure on MZVs, the stuffle product. Its simplest instance reads

$$\zeta_m\;\zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}\,. \tag{3.3}$$

It follows from either eq. (3.2) or eq. (3.3) that the $\mathbb{Q}$-span $\mathcal{Z}$ of all MZVs is a subalgebra of $\mathbb{R}$. Conjecturally, $\mathcal{Z}$ is graded by the weight of the MZVs

$$\mathcal{Z} = \bigoplus_{w=0}^{\infty} \mathcal{Z}_w\,, \tag{3.4}$$

where the dimensions d_w of $\mathcal{Z}_w$ have been conjectured to be $d_w = d_{w-2} + d_{w-3}$ where $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$ [14]. A possible choice of basis elements for each weight w is given in table 3, for higher weights consult ref. [15].

Single ζ-functions of even weight are rather different from their odd-weight counterparts: all single zeta values of even weight $2n$ can be expressed as rational multiples of π^{2n}, which renders them transcendental numbers immediately. For odd single zeta values, however, there is no

w	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{Z}_w$	ζ_2	ζ_3	ζ_2^2	ζ_5 $\zeta_2\ \zeta_3$	ζ_3^2 ζ_2^3	ζ_7 $\zeta_2\ \zeta_5$ $\zeta_2^2\ \zeta_3$	$\zeta_{3,5}$ $\zeta_3\ \zeta_5$ $\zeta_2\ \zeta_3^2$ ζ_2^4	ζ_9 ζ_3^3 $\zeta_2\ \zeta_7$ $\zeta_2^2\ \zeta_5$ $\zeta_2^3\ \zeta_3$	$\zeta_{3,7}$ $\zeta_3\ \zeta_7$ ζ_5^2 $\zeta_2\ \zeta_{3,5}$ $\zeta_2\ \zeta_3\ \zeta_5$ $\zeta_2^2\ \zeta_3^2$ ζ_2^5	$\zeta_{3,3,5}$ $\zeta_{3,5}\ \zeta_3$ ζ_{11} $\zeta_3^2\ \zeta_5$ $\zeta_2\ \zeta_3^3$ $\zeta_2\ \zeta_9$ $\zeta_2^2\ \zeta_7$ $\zeta_2^3\ \zeta_5$ $\zeta_2^4\ \zeta_3$	$\zeta_{1,1,4,6}$ $\zeta_{3,9}$ $\zeta_3\ \zeta_9$ $\zeta_5\ \zeta_7$ ζ_3^4 $\zeta_2\ \zeta_{3,7}$ $\zeta_2^2\ \zeta_{3,5}$ $\zeta_2\ \zeta_5^2$ $\zeta_2\ \zeta_3\ \zeta_7$ $\zeta_2^2\ \zeta_3\ \zeta_5$ $\zeta_2^3\ \zeta_3^2$ ζ_2^6
d_w	1	1	1	2	2	3	4	5	7	9	12

Table 3: A possible choice for the basis elements of $\mathcal{Z}_w$ for $2 \leq w \leq 12$.

analogous property: there are no known relations relating two single zeta values of distinct odd weight, and in fact no such relations are expected. Also, although expected, none of the odd ζ-values has been proven to be transcendental so far: the only known facts are the irrationality of ζ_3 as well as the existence of an infinite number of odd irrational ζ's [34, 35].

3.1 Hopf algebra structure of MZVs

The basis elements in table 3 have been chosen by convenience preferring short and simple ζ's. However, the choice of basis elements does not seem to be intuitive at all, as is exemplified by the appearance of $\zeta_{1,1,4,6}$ at weight 12. It would be desirable to find a language in which one can write down a basis for MZVs in a more transparent way, with all relations built in once the translation is performed. This language does indeed exist: it is furnished by the graded Hopf algebra comodule $\mathcal{U}$, which is composed from words

$$f_{2i_1+1} \dots f_{2i_r+1}\ f_2^k, \quad \text{with} \quad r, k \geq 0 \quad \text{and} \quad i_1, \dots, i_r \geq 1 \tag{3.5}$$

of weight $w = 2(i_1 + \ldots + i_r) + r + 2k$. While words in the letters f_{2i+1} span a Hopf algebra endowed with a commutative shuffle product, the Hopf algebra comodule $\mathcal{U}$ is obtained upon adjoining powers of f_2, which commute with all f_{2i+1} [12]. Writing down all words of the form in eq. (3.5), one indeed finds the dimension of $\mathcal{U}_w$ to match the expected dimension d_w of $\mathcal{Z}_w$, which is a first indicator that the Hopf algebra comodule $\mathcal{U}$ does indeed shed light on the algebraic structure of MZVs.

In a next step MZVs need to be related to elements in $\mathcal{U}$. Unfortunately, due to the difficult problem of excluding algebraic relations between MZVs, this cannot be done directly. In order to circumvent this issue, one lifts MZVs ζ to so-called motivic MZVs ζ^{m}, which have a more elaborate definition [36,12,37], but which still satisfy the same shuffle and stuffle product formulæ as the MZVs eqs. (3.2) and (3.3). Moreover, passing from MZVs to motivic MZVs has the advantage that many of the desirable, but currently unproven facts about MZVs are in fact proven for motivic MZVs. In particular, the commutative algebra $\mathcal{H}$ of motivic multiple zeta values is by definition graded for the weight, and carries a well-defined motivic coaction, first written down by Goncharov [36] and further studied by Brown [12, 11, 37].

With the availability of $\mathcal{H}$ the only remaining piece is the construction of an isomorphism ϕ of graded algebra comodules

$$\phi : \mathcal{H} \to \mathcal{U}\,, \tag{3.6}$$

whose existence is guaranteed by the main result of [12]. The map ϕ, which assigns to each motivic MZV a linear combination of the words defined in eq. (3.5), is thoroughly described and explored in ref. [11]. As pointed out in the reference, the map ϕ is non-canonical and depends on the choice of an algebra basis of $\mathcal{H}$. The requirement that all odd single motivic MZVs as well as ζ_2 should be contained in this basis leads to

$$\phi(\zeta_k^{\mathfrak{m}}) = f_k\,, \quad k = 2, 3, 5, 7, \ldots\,. \tag{3.7}$$

Unfortunately, this convention does not determine ϕ uniquely, since not every motivic MZV can be expressed in terms of motivic single zeta values only. However, as pointed out in ref. [11], the map ϕ preserves all relations between motivic MZVs for any choice of algebra basis of $\mathcal{H}$, for example (cf. eq. (3.3)):

$$\phi(\zeta_m^{\mathfrak{m}}\,\zeta_n^{\mathfrak{m}}) = \phi(\zeta_{m,n}^{\mathfrak{m}}) + \phi(\zeta_{n,m}^{\mathfrak{m}}) + \phi(\zeta_{m+n}^{\mathfrak{m}})\,. \tag{3.8}$$

In order to give explicit examples of the decomposition of motivic MZVs into the f-alphabet, let us choose the following algebra basis up to weight 12, upon which table 3 is modeled implicitly

$$\{\zeta_2^{\mathfrak{m}}, \zeta_3^{\mathfrak{m}}, \zeta_5^{\mathfrak{m}}, \zeta_7^{\mathfrak{m}}, \zeta_{3,5}^{\mathfrak{m}}, \zeta_9^{\mathfrak{m}}, \zeta_{3,7}^{\mathfrak{m}}, \zeta_{11}^{\mathfrak{m}}, \zeta_{3,3,5}^{\mathfrak{m}}, \zeta_{1,1,4,6}^{\mathfrak{m}}, \zeta_{3,9}^{\mathfrak{m}}\}\,. \tag{3.9}$$

With this choice of basis, one finds for example,

$$\begin{aligned}\phi(\zeta_{3,9}^{\mathfrak{m}}) &= -6\,f_5 f_7 - 15\,f_7 f_5 - 27\,f_9 f_3\\ \phi(\zeta_{3,3,5}^{\mathfrak{m}}) &= -5\,f_5 f_3 f_3 + \frac{4}{7}\,f_5 f_2^3 - \frac{6}{5}\,f_7 f_2^2 - 45\,f_9 f_2\,.\end{aligned} \tag{3.10}$$

The application of the ϕ-map in the context of the low-energy expansion of superstring tree-level amplitudes as well as several higher-weight examples can be found in ref. [38].

4 Indecomposable eMZVs, Eisenstein series and the derivation algebra

As described in section 2, indecomposable eMZVs at a certain weight and length can be in principle inferred from considering reflection, shuffle and Fay relations. For higher weights and lengths, however, it is favorable to employ a computer implementation based on comparing q-expansions of eMZVs which in turn can be obtained recursively from eq. (2.47). In this section we are going to provide an algorithm which does not only deliver the appropriate indecomposable elements as listed in table 1 but as well explains their number at a given length and weight.

As described in the previous section, the appropriate mathematical idea for standard motivic MZVs is to map them to the non-commutative words composed from letters f_w in eq. (3.5) using the map ϕ. For the elliptic case we will construct an isomorphism ψ relating the ω-representation of eMZVs to non-commutative words composed from letters g_w, which in turn arise as labels of iterated Eisenstein integrals γ to be defined below.

4.1 Iterated Eisenstein integrals

Given that the q-expansion of eMZVs can be iteratively generated from the Eisenstein series G_k employing eq. (2.47), we will now describe eMZVs based on combinations of G_k. Instead of representing eMZVs as elliptic iterated integrals as in section 2, we will write them as iterated integrals over Eisenstein series G_k where the iterated integration is now performed over the modular parameter τ (or equivalently q).

Iterated integrals over Eisenstein series arise as a subclass of iterated integrals of modular forms, which have been studied in refs. [17,18]. In this section, we will briefly review some of the key definitions in order to embed the subsequent presentation of eMZVs into a broader context.

Iterated integrals of modular forms or *iterated Shimura integrals* [17,18] are defined via

$$\int\limits_{i\infty>\tau_1>\tau_2>\cdots>\tau} \mathrm{d}\tau_1 \, (X_1-\tau_1 Y_1)^{k_1-2}\mathcal{F}_{k_1}(\tau_1)\, \mathrm{d}\tau_2 \, (X_2-\tau_2 Y_2)^{k_2-2}\mathcal{F}_{k_2}(\tau_2)\cdots \\ \cdots \mathrm{d}\tau_n \, (X_n-\tau_n Y_n)^{k_n-2}\mathcal{F}_{k_n}(\tau_n) \, , \tag{4.1}$$

where $\mathcal{F}_k(\tau)$ is a modular form of weight k and the modular group acts on commutative variables X_i and Y_i as to render eq. (4.1) modular invariant. The divergences in these integrals caused by the constant terms in the q-expansion of the modular forms can be regularized in a manner described in ref. [18]. The key idea of this regularization procedure is to separate the constant part from the remaining q-series for each $\mathcal{F}_{k_j}(q)$ and to associate a different integration prescription to it. The mathematical justification of this procedure is furnished by the theory of *tangential base points* [39]. In the present case, one regularizes the integral with respect to the tangential base point $\vec{1}_\infty$ [18].

In the context of eMZVs in eq. (2.11), we encounter special cases of the iterated Shimura integrals defined above, evaluated at $X_i=1$ and $Y_i=0$. Furthermore, the τ-derivative of eMZVs in eq. (2.47) involves no modular forms $\mathcal{F}_k$ other than Eisenstein series G_k. This motivates to study the following *iterated Eisenstein integrals* as building blocks for eMZVs,

$$\gamma(k_1,k_2,\ldots,k_n;q) \equiv \frac{1}{4\pi^2}\int_{0\leq q'\leq q} \mathrm{dlog}\, q' \;\; \gamma(k_1,\ldots,k_{n-1};q')\, \mathrm{G}_{k_n}(q') \\ = \frac{1}{(4\pi^2)^n}\int_{0\leq q_i<q_{i+1}\leq q} \mathrm{dlog}\, q_1 \;\; \mathrm{G}_{k_1}(q_1) \;\; \mathrm{dlog}\, q_2 \;\; \mathrm{G}_{k_2}(q_2) \;\; \ldots \;\; \mathrm{dlog}\, q_n \, \mathrm{G}_{k_n}(q_n) \, , \tag{4.2}$$

where the number n of integrations will be referred to as the *length* ℓ_γ, and the *weight* is given by $w_\gamma = \sum_{i=1}^n k_i$. The definition in eq. (4.2) as an iterated integral immediately implies

$$\frac{\mathrm{d}}{\mathrm{dlog}\, q}\gamma(k_1,k_2,\ldots,k_n;q) = \frac{\mathrm{G}_{k_n}(q)}{4\pi^2}\gamma(k_1,k_2,\ldots,k_{n-1};q) \tag{4.3}$$

$$\gamma(n_1,n_2,\ldots,n_r;q)\,\gamma(k_1,k_2,\ldots,k_s;q) = \gamma\big((n_1,n_2,\ldots,n_r)\sqcup\!\sqcup\,(k_1,k_2,\ldots,k_s);q\big) \, , \tag{4.4}$$

where the dependence on q will be suppressed in most cases: $\gamma(\ldots)\equiv\gamma(\ldots;q)$. The integrals in eq. (4.2) generally diverge due to the constant term in $\mathrm{G}_{k_1} = 2\,\zeta_{k_1} + \mathcal{O}(q)$ and can be regularized using the procedure discussed around eq. (4.7) while preserving eqs. (4.3) and (4.4).

As will be explained in detail below, eMZVs can be expressed in terms of particular linear combinations of iterated Eisenstein integrals in eq. (4.2) such that all possible divergences cancel. An alternative description of eMZVs which manifests the absence of divergences and admits convenient formulæ for their q-expansion will be given in subsection 4.6. The convergent linear combinations of eq. (4.2) occurring in eMZVs will turn out to be governed by a special derivation algebra $\mathfrak{u}$. The situation is summarized in figure 1: eMZVs are a special case of iterated Eisenstein integrals eq. (4.2) which in turn span a subspace of iterated Shimura integrals eq. (4.1).

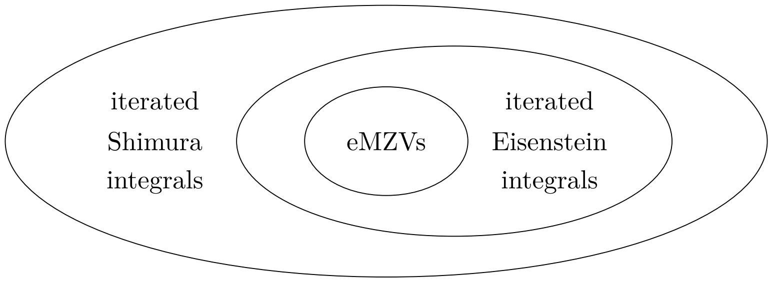

Figure 1: Relation between different type of iterated integrals discussed.

Regularization. Even though eMZVs can be assembled from convergent iterated integrals over modular parameters – see subsection 4.6 – we shall sketch a regularization procedure for the iterated Eisenstein integrals in eq. (4.2) to render individual terms in the subsequent description of eMZVs well-defined. Let us consider the simplest case, namely that of an iterated integral of length one:

$$\gamma(k) = \frac{1}{4\pi^2}\int_0^q \mathrm{G}_k(q')\,\mathrm{dlog}\,q'. \tag{4.5}$$

The term $(\mathrm{G}_k(q') - 2\,\zeta_k)\,\mathrm{dlog}\,q'$ is straightforward to integrate from 0 to q, since it has no poles on the integration domain $0 \leq q' \leq q$. On the other hand, integration of the term $2\,\zeta_k\,\mathrm{dlog}\,q'$ in isolation requires regularization, due to the presence of a simple pole at $q' = 0$. The regularization scheme employed in this case, however, is entirely analogous to the regularization scheme for multiple polylogarithms, MZVs or eMZVs: One introduces a small parameter $\varepsilon > 0$, then expands the integral

$$2\,\zeta_k \int_\varepsilon^q \mathrm{dlog}\,q' = 2\,\zeta_k(\log q - \log\varepsilon) \tag{4.6}$$

as a polynomial in $\log(\varepsilon)$ and finally takes the constant term in this expansion. Using this procedure in the length one case, one obtains from eq. (4.5)

$$\gamma(k) = \frac{1}{4\pi^2}\int_0^q \mathrm{G}_k(q')\,\mathrm{dlog}\,q' = \frac{1}{4\pi^2}\left(2\,\zeta_k\log q + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!}\sum_{m,n=1}^{\infty}\frac{m^{k-2}}{n}q^{mn}\right). \tag{4.7}$$

The regularization procedure for a general iterated Eisenstein integral $\gamma(k_1, k_2, \ldots, k_n)$ as in 4.2 is deduced from the length one case, using the shuffle product formula. Full details can be found in [18].

4.2 eMZVs as iterated Eisenstein integrals

As a first example on how to express eMZVs in terms of iterated Eisenstein integrals, let us consider eq. (2.47) for two simple types of eMZVs (recalling eq. (2.2) and $\mathrm{G}_0 \equiv -1$):

$$2\pi i\frac{\mathrm{d}}{\mathrm{d}\tau}\,\omega(0,n) = -4\pi^2 q\frac{\mathrm{d}}{\mathrm{d}q}\,\omega(0,n) = -2n\,\zeta_{n+1}\,\mathrm{G}_0 - n\,\mathrm{G}_{n+1}, \quad n \text{ odd} \tag{4.8a}$$

$$2\pi i\frac{\mathrm{d}}{\mathrm{d}\tau}\,\omega(0,0,n) = -4\pi^2 q\frac{\mathrm{d}}{\mathrm{d}q}\,\omega(0,0,n) = n\,\omega(0,n+1)\,\mathrm{G}_0\,, \quad n \text{ even}\,. \tag{4.8b}$$

Integration over $\mathrm{dlog}\,q$ relates the eMZVs on the left hand side to iterated Eisenstein integrals of the form eq. (4.2), and the absence of constant terms within τ-derivatives guarantees that the integral converges. This insight will actually be the key ingredient to the simplified representation of eMZVs described in subsection 4.6 below. The rewriting in eqs. (4.8a) and (4.8b) can be generalized for all eMZVs: using the differential equation (2.47) one can represent their derivative as a sum over Eisenstein series G_{2k},

$$\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(n_1,n_2,\ldots,n_r) = \frac{1}{4\pi^2}\sum_{k=0}^{\infty}\xi_{2k}(n_1,n_2,\ldots,n_r)\,\mathrm{G}_{2k}\,, \tag{4.9}$$

where the coefficients $\xi_{2k}(n_1,\ldots,n_r)$ are linear combinations of eMZVs of weight $n_1+\ldots+n_r+1-2k$ and length $r-1$. An example of this decomposition is spelled out below eq. (4.11).

For the eMZVs appearing in the coefficients $\xi_{2k}(n_1,\ldots,n_r)$ of eq. (4.9), the procedure can be repeated to successively reduce the length. Finally, any eMZV can be rewritten in terms of the *iterated Eisenstein integrals* in eq. (4.2). Since the right hand side of eq. (4.9) is a τ-derivative and cannot have a constant term in q, its integral over $\mathrm{d}\log q$ is convergent and the first entries of the resulting iterated Eisenstein integrals for any eMZV are interlocked as $\gamma(k,\ldots)+2\zeta_k\,\gamma(0,\ldots)$.

Examples. Let us return to the examples eqs. (4.8a) and (4.8b). The differential equation eq. (4.3) immediately implies

$$\omega(0,n) = \delta_{1,n}\frac{\pi i}{2} + n\big(\gamma(n+1) + 2\,\zeta_{n+1}\,\gamma(0)\big), \qquad n \text{ odd} \tag{4.10a}$$

$$\omega(0,0,n) = -\frac{1}{3}\,\zeta_n - n(n+1)\big(\gamma(n+2,0) + 2\,\zeta_{n+2}\,\gamma(0,0)\big), \quad n \text{ even}\,, \tag{4.10b}$$

where $\delta_{1,n}\frac{\pi i}{2}$ and $-\frac{1}{3}\,\zeta_n$ arise as integration constants w.r.t. $\log q$. Even though all the above iterated Eisenstein integrals $\gamma(n+1), \gamma(n+2,0), \gamma(0)$ and $\gamma(0,0)$ individually require regularization – see the discussion around eq. (4.7) – any divergence cancels in the linear combinations of schematic form $\gamma(k,\ldots)+2\zeta_k\,\gamma(0,\ldots)$ in eqs. (4.10a) and (4.10b).

The conversion of eMZVs into γ's amounts to recursively applying the differential equation eq. (2.47) and casting it into the form eq. (4.9). At each step, an instance of G_k is separated until one has reached eMZVs of the form in eqs. (4.10a) and (4.10b) exclusively. After converting those into γ's, one reverts the direction and successively integrates using eq. (4.2), supplementing integration constants from eq. (2.43).

Let us demonstrate the conversion into iterated Eisenstein integrals γ for $\omega(0,3,5)$. Employing eq. (2.47), one finds

$$4\pi^2\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(0,3,5) = -15\,\mathrm{G}_4\,\omega(0,5) + 42\,\omega(0,9) + 3\,\omega(4,5)\,, \tag{4.11}$$

i.e. we have $\xi_4(0,3,5) = -15\,\omega(0,5)$ and $\xi_0(0,3,5) = -42\,\omega(0,9) - 3\,\omega(4,5)$ in the notation of eq. (4.9). While $\omega(0,5)$ and $\omega(0,9)$ can be readily converted into γ's using eqs. (4.10a) and (4.10b), we will have to take another derivative[5] for $\omega(4,5)$:

$$\begin{aligned} 4\pi^2\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(4,5) &= 9\,\mathrm{G}_{10}\,\omega(0) - 15\,\mathrm{G}_4\,\omega(6) + 42\,\omega(10) \\ &= 9\,\mathrm{G}_{10} + 30\,\zeta_6\,\mathrm{G}_4 + 84\,\zeta_{10}\,\mathrm{G}_0\,. \end{aligned} \tag{4.12}$$

[5]Alternatively, one could use eq. (2.33), but for illustrational purposes we will perform the recursion explicitly.

Performing the integration eq. (4.2) then leads to

$$\omega(4,5) = 9\,\gamma(10) + 30\,\zeta_6\,\gamma(4) + 84\,\zeta_{10}\,\gamma(0)\,, \tag{4.13}$$

which – after plugged into eq. (4.11) – yields

$$4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\,q}\,\omega(0,3,5) = -75\,\mathrm{G}_4(\gamma(6) + 2\,\zeta_6\,\gamma(0)) + 405\,\gamma(10) + 90\,\zeta_6\,\gamma(4) + 1008\,\zeta_{10}\,\gamma(0)\,. \tag{4.14}$$

After a last integration of the type in eq. (4.2) one finally obtains

$$\omega(0,3,5) = -405\,\gamma(10,0) - 75\,\gamma(6,4) - \zeta_6(150\,\gamma(0,4) + 90\,\gamma(4,0)) - 1008\,\zeta_{10}\,\gamma(0,0)\,, \tag{4.15}$$

which casts the first indecomposable length-three eMZV beyond eq. (4.10b) into the language of iterated Eisenstein integrals and fits into the pattern $\gamma(k,\ldots) + 2\zeta_k\,\gamma(0,\ldots)$ for the first entries. Further examples of expressing eMZVs as iterated Eisenstein integrals are listed in appendix B.2.

Conversion of weight and length. Length and weight are different between the representation of eMZVs in terms of iterated Eisenstein integrals γ and the ω-representation. Denoting length and weight for γ and ω by $(\ell_\gamma,\, w_\gamma)$ and $(\ell_\omega,\, w_\omega)$, respectively, one finds straightforwardly

$$\ell_\gamma = \ell_\omega - 1 \qquad \text{and} \qquad w_\gamma = \ell_\omega - 1 + w_\omega = \ell_\gamma + w_\omega\,, \tag{4.16}$$

such that

$$\gamma(k_1,k_2,\ldots,k_n) \leftrightarrow \text{eMZV in } \omega\text{-rep. with } \ell_\omega = n+1 \text{ and } w_\omega = -n + \sum_{j=1}^{n} k_j \tag{4.17a}$$

$$\omega(n_1,n_2,\ldots,n_r) \leftrightarrow \text{Eisenstein integral with } \ell_\gamma = r-1 \text{ and } w_\gamma = r - 1 + \sum_{j=1}^{r} n_j\;. \tag{4.17b}$$

Those formulæ, however, are valid for the *maximal component* only: as illustrated e.g. in eq. (4.10b), the presentation of eMZVs in terms of iterated Eisenstein integrals involves different lengths ℓ_γ and weights w_γ. Correspondingly, the maximal component is defined to be comprised from all terms in an eMZVs γ-representation, which are of length ℓ_γ and weight w_ω. Below, we will exclude γ's, which can be represented as shuffle products, from the maximal component. Iterated Eisenstein integrals of length $\ell_\gamma - 2, \ell_\gamma - 4, \ldots$ as well as any terms in which weight is carried by MZVs do not belong to the maximal component.

The examples in eq. (4.10b) and eq. (4.15) give rise to maximal components

$$\omega(0,0,n) = -n(n+1)\,\gamma(n+2,0) + \text{non-maximal terms} \tag{4.18}$$

$$\omega(0,3,5) = -405\,\gamma(10,0) - 75\,\gamma(6,4) + \text{non-maximal terms}\;, \tag{4.19}$$

which are defined up to shuffle products of lower-length iterated Eisenstein integrals.

Considering eq. (4.17a), one can create γ's corresponding to ω-representations of negative weight. Since weighting functions $f^{(m)}$ are not defined for negative weight, $\gamma(k_1,k_2,\ldots,k_n)$ with $\sum_{j=1}^{n} k_j < n$ are clearly incompatible with the definition of eMZVs in eq. (2.11). However, the connection with the derivation algebra $\mathfrak{u}$ in subsection 4.3 below will assign a meaning to those γ's in the context of relations between eMZVs at length $\ell_\omega \geq 6$.

Counting of indecomposable eMZVs. What are the advantages of translating eMZVs into iterated Eisenstein integrals? We would like to derive the set of indecomposable eMZVs with given length and weight from purely combinatorial considerations, similar to writing down all non-commutative words of letters f for standard MZVs (cf. eq. (3.5)). In particular, each indecomposable eMZV in table 1 should be related to a particular combination of shuffle-independent γ's. Correspondingly, the counting of indecomposable γ's of appropriate weight and length should be related to the numbers in table 2.

In order to assess the viability of iterated Eisenstein integrals γ for this purpose, it is worthwhile to recall the following observations:

(a) By construction, constant terms are absent in the differential eq. (2.47) for eMZVs. This interlocks the first entries of iterated Eisenstein integrals representing eMZVs in rigid combinations of $\gamma(k,\ldots) + 2\zeta_k\,\gamma(0,\ldots)$. Hence, it is sufficient for counting purposes to focus on $\gamma(k_1,k_2,\ldots,k_r)$ with $k_1 \neq 0$.

(b) The choice of indecomposable eMZVs in table 1 contains no further divergent representative besides $\omega(0,1) = \gamma(2)+2\,\zeta_2\,\gamma(0)+\frac{i\pi}{2}$. For any weight and length considered, divergences in eMZVs are captured by products with $\gamma(2)$ instead of shuffle-irreducible integrals of higher length such as $\gamma(2,4)$. We will assume the continuation of this pattern and confine the choice of labels for all other Eisenstein integrals γ at length $\ell_\gamma \geq 2$ to the set $\{0,4,6,\ldots\}$. This will be justified later on by the observation that the element $\epsilon_2 \in \mathfrak{u}$ corresponding to $\gamma(2)$ is central by eq. (4.27).

(c) The shuffle relations eq. (4.4) allow to reduce various linear combinations of iterated Eisenstein integrals to lower length, e.g.

$$\gamma(4,4) = \frac{1}{2}\,\gamma(4)^2 \quad \text{and} \quad \gamma(6)\,\gamma(4) = \gamma(4,6) + \gamma(6,4)\,, \tag{4.20}$$

and the bookkeeping of indecomposable eMZVs boils down to classifying shuffle-independent Eisenstein integrals γ. At length $\ell_\gamma = 2$ and weight $w_\gamma = 10$, possible indecomposable elements read $\gamma(10,0)$ and $\gamma(6,4)$, because $\gamma(4,6)$ can be obtained using shuffling of γ's of lower length. Similarly, $\ell_\gamma = 2$ and $w_\gamma = 12$ leaves no indecomposable eMZVs beyond $\gamma(12,0)$ and $\gamma(8,4)$.

Let us compare the survey of available Eisenstein integrals with the indecomposable eMZVs in table 1. Eisenstein integrals of length one immediately match with the maximal component of indecomposable eMZVs $\omega(0,2n+1)$ of length two using eq. (4.10a), so the first non-trivial tests occur at length $\ell_\omega = 3$, i.e. $\ell_\gamma = 2$.

Via eq. (4.10b) one finds indeed $\gamma(4,0), \gamma(6,0)$ and $\gamma(8,0)$ to represent the maximal component of $\omega(0,0,2), \omega(0,0,4)$ and $\omega(0,0,6)$, respectively. For $w_\gamma = 10$, which corresponds to $w_\omega = 8$, one can write down two distinct indecomposable elements: $\gamma(10,0)$ and $\gamma(6,4)$. This nicely ties in with the appearance of the second indecomposable eMZV $\omega(0,3,5)$ at $\ell_\omega = 3$ and $w_\omega = 8$, see eqs. (4.18) and (4.19).

Similarly, the aforementioned indecomposable eMZVs $\gamma(12,0)$ and $\gamma(8,4)$ at weight $w_\gamma = 12$ are in concordance with the $w_\omega = 10$ entry of table 1,

$$\begin{aligned}\omega(0,0,10) &= -\frac{\zeta_{10}}{3} - 110\,\gamma(12,0) - 220\,\zeta_{12}\,\gamma(0,0)\\ \omega(0,3,7) &= -294\,\gamma(8,4) - 1848\,\gamma(12,0) + \text{non-maximal terms}\,. \end{aligned} \tag{4.21}$$

The appearance of the indecomposable eMZVs $\omega(0,3,5)$ and $\omega(0,3,7)$ beyond $\omega(0,0,2n)$ matches the existence of shuffle-independent Eisenstein integrals $\gamma(6,4)$ and $\gamma(8,4)$ in addition to $\gamma(10,0)$ and $\gamma(12,0)$.

Surprises from weight-twelve eMZVs and beyond. The literal application of the above reasoning to iterated Eisenstein integrals of weight $w_\gamma = 14$ suggests indecomposable eMZVs

$$\omega(0,0,12) \text{ from } \gamma(14,0), \quad \omega(0,3,9) \text{ from } \gamma(10,4) \quad \text{and} \quad \omega(0,5,7) \text{ from } \gamma(8,6)\,. \tag{4.22}$$

This, however, clashes with the findings noted in table 1: at $\ell_\omega = 3$ and $w_\omega = 12$ we find only *two* indecomposable eMZVs $\omega(0,0,12)$ and $\omega(0,3,9)$, whereas the above counting of appropriate interated Eisenstein integrals would suggest *three* indecomposable eMZVs. In particular, $\omega(0,5,7)$ can be expressed in terms of the two indecomposable eMZVs as written in eq. (2.35).

In order to explain the discrepancy between indecomposable eMZVs and shuffle-independent iterated Eisenstein integrals, let us inspect the first instance at $w_\gamma = 14, \ell_\gamma = 2$, which corresponds to $w_\omega = 12, \ell_\omega = 3$. The natural candidates for indecomposable eMZVs besides $\omega(0,0,12)$ have the following γ-representations,

$$\begin{aligned} \omega(0,3,9) &= -\,315\,\gamma(8,6) - 729\,\gamma(10,4) - 5616\,\gamma(14,0) + \text{non-maximal terms} \\ \omega(0,5,7) &= -\,490\,\gamma(8,6) - 1134\,\gamma(10,4) - 5642\,\gamma(14,0) + \text{non-maximal terms}\,, \end{aligned} \tag{4.23}$$

and the relation eq. (2.35) for $\omega(0,5,7)$ leaves only $\omega(0,0,12)$ and $\omega(0,3,9)$ indecomposable. In general, there seem to be non-obvious restrictions to the Eisenstein integrals γ appearing in eMZVs, beyond the observations (a), (b) and (c). In table 4, we have noted the deviations from the expected pattern at lengths $\ell_\omega \leq 5$. Interestingly, the Eisenstein integrals $\gamma(8,6)$ and $\gamma(10,4)$

ℓ_ω \ w_ω	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		3_{2}		3		4_{3}		4_{3}	
4	1		1		2		3		4		6_{5}		8_{7}		10_{8}		13_{10}		16_{12}
5		1		2		4		6		10_{9}		14_{13}		21_{17}		28_{23}		39_{30}	

Table 4: Number of indecomposable eMZVs at length ℓ according to the counting of $\gamma(k_1,k_2,\ldots,k_n)$ suggested by the above observations (a), (b) and (c). The black numbers denote the number of shuffle-independent γ's with $k_i = 0,4,6,\ldots$ and $k_1 \neq 0$ while the red numbers indicate a deviating number of indecomposable eMZVs found from reflection-, shuffle- and Fay relations or the q-expansion.

enter eq. (4.23) and thus any other eMZV of the same weight and length in the combination

$$35\,\gamma(8,6) + 81\,\gamma(10,4) \tag{4.24}$$

exclusively. The above quantity is the first in a series of links to the derivation algebra $\mathfrak{u}$ introduced and discussed in the next subsection.

4.3 A relation to the derivation algebra $\mathfrak{u}$

The explanation of the deviating numbers for indecomposable eMZVs compared to shuffle-independent Eisenstein integrals in the last subsection can be provided starting from the following differential equation for the KZB associator $A(q)$ defined in eq. (2.37) [1]:

$$\frac{\mathrm{d}}{\mathrm{d}\log q}(e^{i\pi[y,x]}A(q)) = \frac{1}{4\pi^2}\left(\sum_{n=0}^{\infty}(2n-1)\,\mathrm{G}_{2n}(q)\epsilon_{2n}\right)(e^{i\pi[y,x]}A(q))\,. \tag{4.25}$$

The Eisenstein series G_{2n} in eq. (4.25) are accompanied by derivations ϵ_{2n} which act on the non-commutative variables x and y in the expansion of $A(q)$ via

$$\epsilon_{2n}(x) = (\mathrm{ad}_x)^{2n}(y)\ , \qquad n \geq 0 \tag{4.26a}$$

$$\epsilon_{2n}(y) = [y,(\mathrm{ad}_x)^{2n-1}(y)] + \sum_{1\leq j<n}(-1)^j[(\mathrm{ad}_x)^j(y),(\mathrm{ad}_x)^{2n-1-j}(y)]\ , \quad n>0 \tag{4.26b}$$

$$\epsilon_0(y) = 0\,. \tag{4.26c}$$

They generate a Lie subalgebra $\mathfrak{u}$ of the algebra of all derivations on the free Lie algebra generated by x, y [16, 22, 23]. The relations originating from eq. (4.26) have been studied extensively in ref. [24]. Beyond

$$[\epsilon_{2n},\epsilon_2] = 0\ , \quad n \geq 0\ , \tag{4.27}$$

there are several non-obvious relations such as

$$0 = [\epsilon_{10},\epsilon_4] - 3\,[\epsilon_8,\epsilon_6]\,, \tag{4.28a}$$

$$0 = 2\,[\epsilon_{14},\epsilon_4] - 7\,[\epsilon_{12},\epsilon_6] + 11\,[\epsilon_{10},\epsilon_8]\,, \tag{4.28b}$$

$$\begin{aligned}0 = {}&80\,[\epsilon_{12},[\epsilon_4,\epsilon_0]] + 16\,[\epsilon_4,[\epsilon_{12},\epsilon_0]] - 250\,[\epsilon_{10},[\epsilon_6,\epsilon_0]]\\ &- 125\,[\epsilon_6,[\epsilon_{10},\epsilon_0]] + 280\,[\epsilon_8,[\epsilon_8,\epsilon_0]] - 462\,[\epsilon_4,[\epsilon_4,\epsilon_8]] - 1725\,[\epsilon_6,[\epsilon_6,\epsilon_4]]\,.\end{aligned} \tag{4.28c}$$

The rôle of ϵ_2 as a central element in eq. (4.27) is reminiscent of the above observation (b): any appearance of the non-modular G_2 can be captured by powers of $\gamma(2)$. Moreover, a peculiar linear combination of $\gamma(8,6)$ and $\gamma(10,4)$ has been observed in eq. (4.24) to appear in all eMZVs at $\ell_\omega = 3$ and $w_\omega = 12$. Upon identifying labels in γ with those of derivations ϵ_{2n} as suggested by eq. (4.25), one could attribute the selection rule on $\gamma(8,6)$ and $\gamma(10,4)$ to eq. (4.28a).

This connection will be made more precise in the subsequent. For this purpose, iterated Eisenstein integrals will be rewritten in terms of non-commutative letters similar to the ones discussed for usual MZVs in section 3. In particular we are led to an structure reminiscent of the ϕ-map, which provided the key to a convenient representation of MZVs in which all known relations over $\mathbb{Q}$ are automatically built in.

The rewriting of the ω-representation of eMZVs in terms of non-commutative letters turns out to mimick the procedure used in order to define the map ϕ in eq. (3.6). Despite the resemblance, however, the definition of the map ϕ depends on the choice of an algebra basis for motivic MZVs, while the rewriting of eMZVs in terms of non-commutative letters to be described below is completely canonical.

Eisenstein integrals as non-commutative words. As a first step to make the connection between eMZVs and the algebra of derivations manifest, let us translate iterated Eisenstein integrals into words composed from non-commutative generators $g_0, g_2, g_4, \ldots$[6],

$$\psi\big[\gamma(k_1, k_2, \ldots, k_n)\big] \equiv \frac{g_{k_n} g_{k_{n-1}} \cdots g_{k_2} g_{k_1}}{\prod_{j=1}^{n}(k_j - 1)} \,. \tag{4.29}$$

Here, we need to assume that the iterated Eisenstein integrals are linearly independent, and the normalization $g_k/(k-1)$ of the non-commutative alphabet is suggested by the combinations $(k-1)\,\mathrm{G}_k$ in eq. (4.25) and the factors of $n_i\,\mathrm{G}_{n_i+1}$ in eq. (4.10a).

The non-commutative letters g_k are naturally endowed with a shuffle product. The ψ-map defined by eq. (4.29) then satisfies

$$\psi\big[\gamma(n_1, n_2, \ldots, n_r)\gamma(k_1, k_2, \ldots, k_s)\big] = \psi\big[\gamma(n_1, n_2, \ldots, n_r)\big] ш \psi\big[\gamma(k_1, k_2, \ldots, k_s)\big] \,. \tag{4.30}$$

The linear combination of $\gamma(8, 6)$ and $\gamma(10, 4)$ appearing in the eMZVs with $w_\omega = 12$ and $\ell_\omega = 3$ are mapped to

$$\psi\big[35\,\gamma(8, 6) + 81\,\gamma(10, 4)\big] = g_6 g_8 + 3 g_4 g_{10} \,. \tag{4.31}$$

Hence, the image of any $w_\omega = 12, \ell_\omega = 3$ eMZV under eq. (4.29) is annihilated by the differential operator

$$[\partial_{10}, \partial_4] - 3[\partial_8, \partial_6] \,, \tag{4.32}$$

once differentiation of a non-commutative word in g_i is defined to act on the leftmost letter

$$\partial_j g_{k_1} \ldots g_{k_n} = \delta_{j,k_1} g_{k_2} \ldots g_{k_n} \,. \tag{4.33}$$

This differentiation rule satisfies a Leibniz property w.r.t. the shuffle product eq. (4.30) and appeared already in the context of the representation of motivic MZVs in terms of non-commutative letters f_i [11], see the discussion in section 3. Note furthermore that the recursive construction of the eMZVs' ψ-image via eq. (4.9) with coefficients $\xi_{2k}(n_1, \ldots, n_r)$ determined by the differential equation (2.47) is similar to the recursive evaluation of the ϕ-map [11]: The coefficients ξ_{2k+1} of $\phi(\zeta^{\mathfrak{m}}) = \sum_{3 \le 2k+1 \le w} f_{2k+1}\xi_{2k+1}$ for some motivic MZV of weight w are determined by the component of weight $(2k+1) \otimes (w - 2k - 1)$ in the coaction. Hence, the τ-derivative in the form eq. (4.9) exhibits a formal similarity to the coaction of motivic MZVs.

However, there is an important difference between the ϕ-map and the rewriting of eMZVs in its ψ-image: while the ϕ-map depends on a choice of algebra generators (for example the adaptation of eq. (3.10) to the basis in table 3), the ψ-map for eMZVs is completely canonical.

In summary, the ψ-image of an eMZV $\omega(n_1, \ldots, n_r)$ is computed in two steps:

- use the differential equation to write $\omega(n_1, \ldots, n_r)$ as a linear combination of iterated Eisenstein integrals $\gamma(k_1, \ldots, k_s)$. Relying on our working hypothesis that iterated Eisenstein integrals are linearly independent, this decomposition is unique.

- apply the map in eq. (4.29) to each of the γ's.

Non-commutative differentiation and the derivation algebra $\mathfrak{u}$. The similarity between eqs. (4.28a) and (4.32) suggests to identify derivations ϵ_{2m} with derivatives with respect to the non-commutative letters ∂_{2m}. Indeed, we will verify in three steps that the derivations ϵ_{2m}

[6]We are grateful to Francis Brown who helped us to understand the language and scope of non-commutative words in the context of multiple modular values, in particular for pointing us to section 12 of ref. [18].

encode the action of ∂_{2m} on the ψ-image of the KZB associator eq. (2.37) and therefore on the ψ-image of any eMZVs:

(i) integrate the differential equation (4.25) of the KZB associator,

$$e^{i\pi[y,x]}(A(q)-A(0)) = e^{i\pi[y,x]}\frac{1}{4\pi^2}\sum_{n=0}^{\infty}(2n-1)\int_0^q \mathrm{dlog}\, q' \;\; \mathrm{G}_{2n}(q')\epsilon_{2n}A(q') \ , \tag{4.34}$$

using the corollary $\epsilon_{2n}([y,x])=0$ of eq. (4.26) to commute $\epsilon_{2n}e^{i\pi[y,x]} = e^{i\pi[y,x]}\epsilon_{2n}$

(ii) apply the ψ-map defined in eq. (4.29):

$$\psi[A(q)-A(0)] = \sum_{n=0}^{\infty}\epsilon_{2n}g_{2n}\psi[A(q)] \ , \tag{4.35}$$

using the fact that integration against $\frac{(2n-1)}{4\pi^2}\mathrm{G}_{2n}$ amounts to left-concatenation with g_{2n}

(iii) act with ∂_{2m} such that the sum over n collapses by eq. (4.33),

$$\partial_{2m}\psi[A(q)-A(0)] = \partial_{2m}\psi[A(q)] = \partial_{2m}\sum_{n=0}^{\infty}\epsilon_{2n}g_{2n}\psi[A(q)] = \epsilon_{2m}\psi[A(q)] \ , \tag{4.36}$$

where we used that the derivative ∂_{2m} annihilates the boundary term $A(0)$, which translates into an empty word in the letters g.

This is the reason, why any relation among the derivations ϵ_i defines a differential operator via $\epsilon_i \to \partial_i$ which annihilates the ψ-image of any eMZV. Explicitly:

$$\forall \ E\in\mathfrak{u} \text{ such that } E(x)=E(y)=0 \quad \Rightarrow \quad E\big|_{\epsilon_{2m}\to\partial_{2m}}\psi[\omega(n_1,\ldots,n_r)] = 0 \ . \tag{4.37}$$

Thus, any relation in $\mathfrak{u}$ obstructs the appearance of some single linear combination of iterated Eisenstein integrals eq. (4.2) among eMZVs and reduces the counting of indecomposable representatives at lengths and weights governed by the conversion rules eq. (4.17a).

4.4 Systematics of relations in the derivation algebra

Naturally, we have been checking the implications of counting shuffle-independent $\gamma(k_1,k_2,\ldots,k_n)$ subject to $k_1\neq 0$ and $k_i\neq 2$ (cf. the three observations around eq. (4.20)) and the connection with the derivation algebra $\mathfrak{u}$ established in the previous subsection by comparing q-expansions: up to weights $w_\omega = 30,18,8,5$ for $\ell_\omega = 3,4,5,6$ we find complete agreement. There are, however, no obstructions for repeating the analysis for eMZVs of higher length, as tested for several low weights at length 7 and 8.

Counting relations from the algebra of derivations $\mathfrak{u}$ for a given weight and depth works as follows: we start with an ansatz for a relation E of the form

$$0 \stackrel{!}{=} \sum_{\{n_1,n_2,\ldots,n_r\}}\alpha_{n_1,n_2,\ldots,n_r}[[\ldots[[\partial_{n_1},\partial_{n_2}],\partial_{n_3}],\ldots],\partial_{n_r}] \tag{4.38}$$

with rational fudge coefficients $\alpha_{n_1,n_2,\ldots,n_r}$ and $\{n_1,n_2,\ldots,n_r\}$ composed of $n_i = 0,4,6,\ldots$ of appropriate weight and length. The number r of partial derivatives in the nested commutators of eq. (4.38) (or the number of ϵ_n in the dual derivations, respectively) is referred to as *depth*. Of course, the summation in eq. (4.38) is restricted to nested commutators which are independent under Jacobi identities.

Considering eq. (4.37), the above ansatz for E should annihilate all ψ-images of eMZVs of the length and weight considered. Using a sufficiently large set of eMZVs, one can easily fix all fudge coefficients in the ansatz and thus extract relations.

Using this method, we find perfect agreement of eq. (4.38) as an operator equation acting on eMZVs with the relations in the derivation algebra available in refs. [24, 40, 41]. In the following paragraphs we will review their classification and extend the explicit results to higher commutator-depths.

Special rôle of ϵ_2. As already observed above, none of the indecomposable eMZVs besides $\omega(0,1)$ does contain an Eisenstein integral involving G_2. This reflects the rôle of ϵ_2 as a central element, as noted in eq. (4.27). Hence, it is sufficient to study commutator relations without ϵ_2.

Irreducible versus reducible relations. Any relation in the derivation algebra $\mathfrak{u} \ni E = 0$ of the form eq. (4.38) yields an infinity of higher-depth corollaries by repeated adjoint action of ϵ_n:

$$E = 0 \quad \Rightarrow \quad \mathrm{ad}_{n_1,n_2,\ldots,n_k}(E) \equiv [\epsilon_{n_1}, [\epsilon_{n_2}, [\ldots, [\epsilon_{n_k}, E]\ldots]]] = 0 \; . \tag{4.39}$$

Any instance of eq. (4.39) with $k > 0$ and E denoting a vanishing combination of ϵ_n-commutators is called a *reducible relation*, whereas relations that cannot be cast into the form $\mathrm{ad}_{n_1,n_2,\ldots,n_k}(E) = 0$ are referred to as *irreducible*. For instance, the simplest non-obvious relation eq. (4.28a) is irreducible and gives rise to reducible relations such as

$$[\epsilon_n, [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6]] = 0 \, , \tag{4.40}$$

and generalizations to higher depth. They affect the bookkeeping of irreducible eMZVs starting from $w_\gamma = 14$ and $\ell_\gamma = 3$, which corresponds to $w_\omega = 11$ and $\ell_\omega = 4$.

A correspondence between cusp forms of weight w and irreducible relations at depth d and weight $w+2(d-1)$ has been discussed in ref. [24]. In the same way as the number of cusp forms at modular weight w is given by

$$\chi_w \equiv \begin{cases} \lfloor \frac{w}{12} \rfloor - 1 & : \; w = 2 \bmod 12 \\ \lfloor \frac{w}{12} \rfloor & : \; \text{other even values of } w \end{cases} \; , \tag{4.41}$$

we expect $\chi_{w-2(d-1)}$ irreducible relations at weight w and depth d relevant to eMZVs of non-negative weight w_ω (see eq. (4.16) for its relation to the weight of the iterated Eisenstein integral). In table 5, this conjectural counting is exemplified up to $w_\gamma = 30$ with a notation $r^d_{w_\gamma}$ for such irreducible relations. Relations of depth two can be cast into a closed formula [26]

$$0 = \sum_{i=1}^{2n+2p-1} \frac{[\epsilon_{2p+2n-i+1}, \epsilon_{i+1}]}{(2p+2n-i-1)!} \left\{ \frac{(2n-1)! B_{i-2p+1}}{(i-2p+1)!} + \frac{(2p-1)! B_{i-2n+1}}{(i-2n+1)!} \right\} , \tag{4.42}$$

where $p, n \geq 1$ denote arbitrary integers and B_n are Bernoulli numbers. Each term of eq. (4.42) carries weight $2(p+n+1)$, e.g. the weight-14 relation eq. (4.28a) follows from any partition of $p+n=6$, and the weight-18 relation eq. (4.28b) from any partition of $p+n=8$.

Irreducible relations at higher depth can be obtained in electronic form from the website `https://tools.aei.mpg.de/emzv`, whereas relations of depth three at $w = 16, 20$ and depth four at $w = 18, 22$ are provided in ref. [24]. New relations beyond those in said reference are

w_γ \ ℓ_γ	2	3	4	5	6	7	8	9	10
12	0	0	0	0	0	0	0	0	0
14	r^2_{14}	0	0	0	0	0	0	0	0
16	0	r^3_{16}	0	0	0	0	0	0	0
18	r^2_{18}	0	r^4_{18}	0	0	0	0	0	0
20	r^2_{20}	r^3_{20}	0	r^5_{20}	0	0	0	0	0
22	r^2_{22}	r^3_{22}	r^4_{22}	0	r^6_{22}	0	0	0	0
24	r^2_{24}	r^3_{24}	r^4_{24}	r^5_{24}	0	r^7_{24}	0	0	0
26	$2 \times r^2_{26}$	r^3_{26}	r^4_{26}	r^5_{26}	r^6_{26}	0	r^8_{26}	0	0
28	r^2_{28}	$2 \times r^3_{28}$	r^4_{28}	r^5_{28}	r^6_{28}	r^7_{28}	0	r^9_{28}	0
30	$2 \times r^2_{30}$	r^3_{30}	$2 \times r^4_{30}$	r^5_{30}	r^6_{30}	r^7_{30}	r^8_{30}	0	r^{10}_{30}

Table 5: Irreducible relations r^ℓ_w. Up to weight 30 there are no more than two relations at a particular weight and length, which will, however, change proceeding to higher weight and length. An actual list of the first irreducible relations is available in appendix C.1.

obtained from the differential operators eq. (4.38) annihilating all eMZVs of corresponding weight and length. This approach to finding relations in the derivation algebra appears computationally more efficient to us than evaluating the action of elements of the derivation algebra on generators x and y of the free Lie algebra. However, once a candidate relation has been identified, it is straightforward to check its validity using its action on the letters x and y via eq. (4.26).

Vanishing nested commutators. Starting from $w_\gamma = 8$ and $\ell_\gamma = 5$, we find that the ψ-image of any eMZV with appropriate weight and length is annihilated by operators of the form

$$[[[[\partial_4, \partial_0], \partial_0], \partial_0], \partial_{2m}] \ . \tag{4.43}$$

The reason becomes clear by considering $\gamma(4,0,0,0)$, one of the corresponding Eisenstein integrals. By eq. (4.16), related eMZVs are bound to have $\ell_\omega = 5$ and $w_\omega = 0$, but the only eMZV with these properties is $\omega(0,0,0,0,0) = 1/120$ which cannot equal the non-constant $\gamma(4,0,0,0)$. Hence, the latter does not occur among eMZVs and signals the irreducible relation

$$[[[\epsilon_4, \epsilon_0], \epsilon_0], \epsilon_0] = 0 \ , \tag{4.44}$$

which in turn implies that $[[[\partial_4, \partial_0], \partial_0], \partial_0]$ annihilates the KZB associator by eq. (4.36). The relation eq. (4.44) can be understood from the organization of $\mathfrak{u}$ in terms of representations of the Lie algebra $\mathfrak{sl}_2$: considering ϵ_{2m} as the lowest-weight state in a $(2m-1)$-dimensional module, the highest-weight vector $\mathrm{ad}_0^{2m-2}\epsilon_{2m}$ is annihilated by further adjoint action of ϵ_0.

Further irreducible relations of this type include

$$\mathrm{ad}_0^{p-1}\epsilon_p = \underbrace{[\epsilon_0, \dots [\epsilon_0, [\epsilon_0,}_{p-1 \text{ times}} \epsilon_p]] \dots] = 0 \ , \qquad p = 4, 6, 8, \dots \ , \tag{4.45}$$

corresponding to the Eisenstein integral $\gamma(p, 0^{p-1})$ with would-be eMZV partners of vanishing

w_ω. Different partitions of the weight in eq. (4.45) lead to further relations such as

$$[[[[[[\epsilon_4,\epsilon_0],\epsilon_4],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0]=0\ ,\qquad [[[[[[[[\epsilon_4,\epsilon_0],\epsilon_6],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0]=0\ . \tag{4.46}$$

Since all their permutations via $\epsilon_4 \leftrightarrow \epsilon_0$ or $\epsilon_6 \leftrightarrow \epsilon_0$ can be identified as a reducible relation descending from eq. (4.45), we expect no further irreducible relations at $d = w_\gamma = 8$ or 10 besides eq. (4.46).

Additional generators of the Lie algebra Consider the free Lie algebra $\mathfrak{k} = \mathbb{L}(z_3, z_5, z_7, \ldots)$ generated by one element in every odd degree strictly greater than one. As mentioned on page 6 of ref. [24], every generator z_{2k+1} of $\mathfrak{k}$ defines a derivation $\widetilde{z}_{2k+1}$ of depth $2k+1$ and weight $4k+2$ of the free Lie algebra on two generators x, y, and satisfies $[\widetilde{z}_{2k+1}, \mathfrak{u}] \subset \mathfrak{u}$. More precisely, the elements ϵ_0, ϵ_2 are annihilated by the elements $\widetilde{z}_{2k+1}$

$$0 = [\widetilde{z}_{2k+1}, \epsilon_0] = [\widetilde{z}_{2k+1}, \epsilon_2]\ , \qquad k = 1, 2, 3, \ldots\ , \tag{4.47}$$

and their commutators with $\epsilon_4, \epsilon_6, \ldots$ can be constructed using the techniques of [24], e.g.

$$[\widetilde{z}_3, \epsilon_4] = -\frac{1}{14}[[\epsilon_4,\epsilon_0],[\epsilon_6,\epsilon_0]] + \frac{1}{42}[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_6]]] - \frac{1}{7}[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_4]]]\,. \tag{4.48}$$

They give rise to further reducible relations, starting from length five at weights $20, 24, 26, \ldots$ by the commutator of $\widetilde{z}_3$ with the depth-two relations in eq. (4.28) or eq. (4.42).

4.5 Counting relations between nested commutators

Example. In order to demonstrate the virtue of the derivation algebra as a counting formalism for indecomposable eMZVs, let us consider $w_\gamma = 20, \ell_\gamma = 5$ as a specific example, which corresponds to $w_\omega = 15, \ell_\omega = 6$. This is the first situation, where all four types of relations described in the previous section have to be taken into account in order to arrive at what we believe is the correct counting of eMZVs.

The naïve enumeration of shuffle-independent γ's with $k_1 \neq 0$ and $k_i \neq 2$ leads to 55 distinct elements. Each relation of depth 5 and weight 20 in the derivation algebra will lower this number according to eq. (4.37).

Let us first consider reducible relations. Starting from table 5, one can construct the following reducible relations by adjoint action of ϵ_n (recalling the notation $r^d_{w_\gamma}$ for irreducible relations of depth d and weight w_γ as well as $\mathrm{ad}_{n_1,n_2,\ldots,n_k} r^j_i \equiv [\epsilon_{n_1},[\epsilon_{n_2},[\ldots,[\epsilon_{n_k}, r^d_{w_\gamma}]\ldots]]]$):

$$\begin{array}{rcl} \mathrm{ad}_{6,0,0}\, r^2_{14} & \leftrightarrow & 3 \text{ permutations}\ , \\ \mathrm{ad}_{4,0}\, r^3_{16} & \leftrightarrow & 2 \text{ permutations}\ , \end{array} \qquad \begin{array}{rcl} \mathrm{ad}_{0,0,0}\, r^2_{20} & \leftrightarrow & 1 \text{ permutation} \\ \mathrm{ad}_{0,0}\, r^3_{20} & \leftrightarrow & 1 \text{ permutation} \end{array}\ . \tag{4.49}$$

In addition, there is one relation each descending from the vanishing nested commutator eq. (4.44) and the additional Lie algebra generator $\widetilde{z}_3$,

$$[[[[\epsilon_4,\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_{16}] = 0 \quad \text{and} \quad [\widetilde{z}_3, r^2_{14}] = 0\,, \tag{4.50}$$

which makes a total of 9 reducible relations.

Indeed, starting with an ansatz of the form eq. (4.38), we find ten distinct relations: while eqs. (4.49) and (4.50) are confirmed, our method explicitly delivers the new irreducible relation r^5_{20} expected from table 5. To our knowledge this is the first appearance of an explicit relation at depth 5 in $\mathfrak{u}$, which is written out in appendix C.5. Correspondingly, we find the number of indecomposable eMZVs at $(\ell_\gamma, w_\gamma) = (5, 20)$ (or $(\ell_\omega, w_\omega) = (6, 15)$) to be 45.

General. In order to repeat the counting procedure from the above example for a variety of weights and lengths, the following tables give an overview of the required ingredients: The numbers of shuffle-independent iterated Eisenstein integrals compatible with observations (a) and (b) in subsection 4.2 are gathered in table 6 and have to be compared with the counting of relations in $\mathfrak{u}$ seen in table 7. Once the offset between (w_γ, ℓ_γ) and (w_ω, ℓ_ω) in eq. (4.17a) is taken into account, one arrives at the numbers of indecomposable eMZVs in the ω-representation noted in table 8.

ℓ_γ \ w_γ	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	1	1	1	2	2	3	3	4	4	5	5	6	6	7
3	0	0	1	1	2	3	4	6	8	10	13	16	19	23	27	31
4	0	0	1	1	2	4	6	10	14	21	28	39	50	66	82	104
5	0	0	1	1	3	5	9	15	24	37	55	80	113	156	211	280
6	0	0	1	1	3	6	11	21	35	59	93	146	217	322	459	649
7	0	0	1	1	4	7	15	28	51	89	150	245	389	602	910	1347

Table 6: Shuffle-independent $\gamma(k_1, \ldots, k_n)$ subject to $k_1 \neq 0$ and $k_i \neq 2$ at various weights w_γ and lengths ℓ_γ.

d \ w_γ	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
2	0	0	0	0	0	0	1	0	1	1	1	1	2	1	2	2	2	2	3	2
3	0	0	0	0	0	0	1	1	2	3	4	5	7	8	10	12	14	16	19	21
4	0	1	0	0	0	0	1	1	4	5	9	13	19	?	?	?	?	?	?	?
5	0	1	0	1	1	1	2	2	6	10	?	?	?	?	?	?	?	?	?	?
6	0	1	1	2	2	3	5	6	11	?	?	?	?	?	?	?	?	?	?	?

Table 7: Relations in the derivation algebra at various weights w_γ and depths d, excluding the central element ϵ_2.

From the above data, one readily arrives at all-weight statements on the number of indecomposable eMZVs of length $\ell_\omega \leq 4$:

- At length $\ell_\omega = 2$, there is obviously one indecomposable eMZV at each odd weight w_ω.
- At length $\ell_\omega = 3$, the number of indecomposable eMZVs at even weight w_ω is $\lceil \frac{1}{6} w_\omega \rceil$. This follows from comparing the number $\lceil \frac{w_\omega}{4} \rceil - 1$ of admissible $\gamma(k_1, k_2)$ $(k_1 > k_2,\ k_i \neq 2)$ at weight $w_\omega > 4$ with the counting of depth-two relations in $\mathfrak{u}$ governed by eq. (4.41).
- At length $\ell_\omega = 4$, the number of indecomposable eMZVs at odd weight w_ω is conjectured to be $\lfloor \frac{1}{2} + \frac{1}{48}(w_\omega + 5)^2 \rfloor$. This conjecture stems from extrapolating [42] the data available at $w_\omega \leq 37$. The extrapolation will remain valid, if the counting of irreducible r_w^3 keeps on following the cusp forms.

ℓ_ω \ w_ω	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	1		1		1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3		4		4	
4	1		1		2		3		4		5		7		8		10		12		14		16
5		1		2		4		6		9		13		17		23		30		37		47	
6	1		2		4		8		13		22		31		45		?		?		?		?
7		1		4		8		16		29		48		?		?		?		?		?	

Table 8: Numbers $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs in their ω-representation. This is an extended version of table 2, where the black results are obtained by explicitly determining q-expansions while results printed in blue originate from testing relations between nested commutators as described around eq. (4.38).

Starting from the next length, $\ell_\omega = 5$ or $\ell_\gamma = 4$, an effect well-known from the algebra of MZVs kicks in: because the lowest non-trivial relation from the derivation algebra $\mathfrak{u}$ exists at weight 14 depth 2, there is the possibility to obtain the "relation of a relation" $\mathrm{ad}_{r^2_{14}}(r^2_{14}) = 0$ at weight 28, depth 4. This effect, which appears in iterated form for higher depth, as well as the action of the generators of the free Lie algebra $\mathfrak{k}$ described in subsection 4.4 render the counting at higher depth difficult. Correspondingly, a closed formula, e.g. a generating series for the number of indecomposable eMZVs at given length and weight is still lacking and some of the entries in table 7 are left undetermined.

4.6 A simpler representation of the eMZV subspace

From the discussion in the previous subsections it became clear that eMZVs can be nicely represented in terms of iterated Eisenstein integrals eq. (4.2). While those integrals have to be regularized individually as pointed out in the context of eq. (4.7), the representation of eMZVs cannot involve any divergences upon integrating their τ-derivative eq. (2.47). In this section we would like to manifest this property and define a modified version of iterated Eisenstein integrals γ_0, which are individually convergent by construction. By using the γ_0-language, one will trade some of the connections to periods and motives [18] inherent in the γ-language for compactness of representation. A further advantage of the γ_0-language to be introduced is a better accessibility of the q-expansions of eMZVs.

Modified iterated Eisenstein integrals. Already in subsection 4.2 it was remarked that the τ-derivative of eMZVs determined by the differential equation (2.47) cannot contain any constant terms. Therefore, it is an obvious idea to subtract the constants from the non-trivial Eisenstein series before defining their iterated integrals:

$$\begin{aligned} \mathrm{G}^0_0 &\equiv -1 \\ \mathrm{G}^0_k &\equiv \mathrm{G}_k - 2\,\zeta_k = \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn}\,, \qquad k \text{ even},\ k \neq 0\,. \end{aligned} \tag{4.51}$$

Using this definition, one can rewrite eqs. (4.8a) and (4.8b) as

$$\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(0,n) = \frac{n}{4\pi^2}\,\mathrm{G}^0_{n+1}, \qquad n \text{ odd} \tag{4.52a}$$

$$\frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \omega(0,0,n) = \frac{n}{4\pi^2}\, \omega(0,n+1)\, \mathrm{G}_0^0, \qquad n \text{ even },\tag{4.52b}$$

and the differential equation (2.47) for generic eMZVs can be easily cast into the form

$$\frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \omega(n_1,n_2,\ldots,n_r) = \frac{1}{4\pi^2} \sum_{k=0}^{\infty} \rho_{2k}(n_1,n_2,\ldots,n_r)\, \mathrm{G}_{2k}^0 \; .\tag{4.53}$$

In complete analogy to eq. (4.9), the coefficients $\rho_{2k}(n_1,\ldots,n_r)$ are linear combinations of eMZVs with weight $n_1+\ldots+n_r+1-2k$ and length $r-1$, the only difference being that Eisenstein series in eq. (2.47) are now expanded via $\mathrm{G}_k = \mathrm{G}_k^0 - 2\,\zeta_k\, \mathrm{G}_0^0$ whenever $k \neq 0$.

From the form eq. (4.53) of the differential eq. (2.47), it is straightforward to introduce modified iterated Eisenstein integrals γ_0 via

$$\begin{aligned}
\gamma_0(k_1,k_2,\ldots,k_n;q) &\equiv \frac{1}{4\pi^2} \int_{0\le q'\le q} \mathrm{dlog}\, q' \;\; \gamma_0(k_1,\ldots,k_{n-1};q')\, \mathrm{G}_{k_n}^0(q') \; , \qquad k_1 \neq 0 \\
&= \frac{1}{(4\pi^2)^n} \int_{0\le q_i<q_{i+1}\le q} \mathrm{dlog}\, q_1 \;\; \mathrm{G}_{k_1}^0(q_1) \;\; \mathrm{dlog}\, q_2 \;\; \mathrm{G}_{k_2}^0(q_2) \;\; \ldots \;\; \mathrm{dlog}\, q_n\, \mathrm{G}_{k_n}^0(q_n) \; ,
\end{aligned}\tag{4.54}$$

whose definition as an iterated integral implies

$$\frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \gamma_0(k_1,k_2,\ldots,k_n;q) = \frac{\mathrm{G}_{k_n}^0(q)}{4\pi^2}\, \gamma_0(k_1,k_2,\ldots,k_{n-1};q)\tag{4.55}$$

$$\gamma_0(n_1,n_2,\ldots,n_r;q)\, \gamma_0(k_1,k_2,\ldots,k_s;q) = \gamma_0\big((n_1,n_2,\ldots,n_r) ш (k_1,k_2,\ldots,k_s);q\big) \, .\tag{4.56}$$

The notion of weight and length are not altered w.r.t. the definition for γ. Naturally, γ_0's suffer from the same caveat with respect to linear independence as their cousins γ. There are several advantages of employing this modified class of iterated Eisenstein integrals γ_0 for the description of eMZVs:

- Logarithmic divergences for $q \to 0$ as present in eq. (4.2) do not occur after setting $k_1 \neq 0$. Modified iterated Eisenstein integrals γ_0 are perfectly well-defined objects which do not require regularization.
- The number of terms necessary to express eMZVs as combinations of iterated Eisenstein integrals γ_0 is significantly lower than for γ.
- The absence of constant terms in the expansion of $\mathrm{G}_{k_1}^0$ propagates to any convergent iterated Eisenstein integral,
$$\gamma_0(k_1,k_2,\ldots,k_n;0) = 0 \; .\tag{4.57}$$

Note that we will again suppress the dependence on q in most cases: $\gamma_0(\ldots) \equiv \gamma_0(\ldots;q)$.

Examples. Let us return to the examples eqs. (4.52a) and (4.52b). The differential equation (4.55) immediately implies

$$\begin{aligned}
\omega(0,n) &= \delta_{1,n} \frac{\pi i}{2} + n\, \gamma_0(n+1), \qquad && n \text{ odd} \qquad (4.58a)\\
\omega(0,0,n) &= -\frac{1}{3}\, \zeta_n - n(n+1)\, \gamma_0(n+2,0), \qquad && n \text{ even}\, , \qquad (4.58b)
\end{aligned}$$

where $\delta_{1,n}\frac{\pi i}{2}$ and $-\frac{1}{3}\,\zeta_n$ arise as integration constants w.r.t. $\log q$. Indeed, these expressions are convergent by definition and shorter than their counterparts in eqs. (4.10a) and (4.10b).

For illustrational purposes let us also revisit the example $\omega(0,3,5)$. Its derivative

$$4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \omega(0,3,5) = 30\,\zeta_6\, \omega(0,3) - 15(\mathrm{G}_4^0 - 2\,\zeta_4\, \mathrm{G}_0^0)\, \omega(0,5) + 45\, \omega(0,9) \tag{4.59}$$

amounts to $\rho_4(0,3,5) = -15\,\omega(0,5)$ and $\rho_0(0,3,5) = 30\,\zeta_4\,\omega(0,5) - 45\,\omega(0,9) - 30\,\zeta_6\,\omega(0,3)$ in the notation of eq. (4.9) and can be translated to modified Eisenstein integrals via eq. (4.58a):

$$4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \omega(0,3,5) = 90\,\zeta_6\, \gamma_0(4) - 75\,(\mathrm{G}_4^0 - 2\,\zeta_4\, \mathrm{G}_0^0)\, \gamma_0(6) + 405\, \gamma_0(10) \tag{4.60}$$

Integration using eq. (4.54) yields the following alternative representation to eq. (4.15),

$$\omega(0,3,5) = -90\,\zeta_6\, \gamma_0(4,0) + 150\,\zeta_4\, \gamma_0(6,0) - 75\, \gamma_0(6,4) - 405\, \gamma_0(10,0)\,. \tag{4.61}$$

Further examples of eMZVs expressed in the language of modified iterated Eisenstein integrals can be found in appendix B.

***q*-expansion.** In contrast to the γ-language used in the last section, there is no caveat on regularization when performing the integrals over q_j in the definition eq. (4.54) of γ_0. The q-expansion stems from the expression for Eisenstein series in eq. (4.51) and can be cast into a closed form (with 0^n denoting a sequence of n entries $0,0,\ldots,0$):

$$\begin{aligned}\gamma_0(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}) &= \prod_{j=1}^{r} \Big(-\frac{2(2\pi i)^{k_j - 2p_j}}{(k_j-1)!} \Big) \\ &\times \sum_{m_i,n_i=1}^{\infty} \frac{m_1^{k_1-1} m_2^{k_2-1} \ldots m_r^{k_r-1} q^{m_1n_1 + m_2n_2 + \ldots + m_rn_r}}{(m_1n_1)^{p_1}(m_1n_1+m_2n_2)^{p_2} \ldots (m_1n_1+m_2n_2+\ldots+m_rn_r)^{p_r}}\,.\end{aligned} \tag{4.62}$$

An even more compact representation can be achieved using the divisor sum

$$\sigma_k(n) \equiv \sum_{d|n} d^k\,, \tag{4.63}$$

which allows to rewrite eq. (4.62) as

$$\begin{aligned}\gamma_0(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}) &= \prod_{j=1}^{r} \Big(-\frac{2(2\pi i)^{k_j - 2p_j}}{(k_j-1)!} \Big) \\ &\times \sum_{0<n_1<n_2<\cdots<n_r} \frac{\sigma_{k_1-1}(n_1)\sigma_{k_2-1}(n_2-n_1)\ldots\sigma_{k_r-1}(n_r-n_{r-1}) q^{n_r}}{n_1^{p_1} n_2^{p_2} \ldots n_r^{p_r}}\,.\end{aligned} \tag{4.64}$$

The above expression bears some resemblance to the sum representation eq. (2.1) of MZVs. One could wonder if rearrangements of the sums could yield a genus-one analogue of stuffle relations. However, both the appearance of the divisor sums and the q-dependence prevent such manipulations. In fact, we did not observe a single relation among iterated Eisenstein integrals γ_0 beyond the shuffle relations eq. (4.4) up to weights $44, 31, 22, 19$ for length $2,3,4,5$, respectively.

Given the above γ_0-representation of the simplest eMZVs, we arrive at two closed forms for

q-expansions

$$\omega(0,k) = \delta_{k,1}\frac{i\pi}{2} + \frac{2(-1)^{(k+1)/2}(2\pi)^{k-1}}{(k-1)!}\sum_{m,n=1}^{\infty}\frac{m^{k-1}}{n}q^{mn}\ , \qquad k \text{ odd} \tag{4.65}$$

$$\omega(0,0,k) = -\frac{1}{3}\zeta_k + \frac{2(-1)^{(k+2)/2}(2\pi)^{k-2}}{(k-1)!}\sum_{m,n=1}^{\infty}\frac{m^{k-1}}{n^2}q^{mn}\ , \qquad k \text{ even}\,, \tag{4.66}$$

while further expressions for interesting $\omega(0,0,\ldots,0,k)$ at higher length are given in appendix B.1.

Connection with the derivation algebra. A manifestly convergent description of eMZVs in terms of modified iterated Eisenstein integrals γ_0 comes with a price at the end of the day: the constant terms which have been omitted in the definition (4.54) have to be restored in order to establish a connection with the derivation algebra. In particular, the translation of modified iterated Eisenstein integrals into the language of non-commutative words built from letters g_k described in subsection 4.3 involves various shifts $\sim \zeta_k\, g_0$,

$$\begin{aligned}\psi\big[\gamma_0(k_1,0^{p_1},k_2,0^{p_2},\ldots,k_n,0^{p_n})\big] &= (-1)^{p_1+p_2+\ldots+p_n} \\ \times\ (g_0)^{p_n}\Big(\frac{g_{k_n}}{k_n-1} - 2\,\zeta_{k_n}\,g_0\Big)&\cdots(g_0)^{p_2}\Big(\frac{g_{k_2}}{k_2-1} - 2\,\zeta_{k_2}\,g_0\Big)(g_0)^{p_1}\Big(\frac{g_{k_1}}{k_1-1} - 2\,\zeta_{k_1}\,g_0\Big)\,,\end{aligned} \tag{4.67}$$

where $(g_0)^n$ refers to n adjacent letters g_0. Furthermore, the concatenation of words is understood to act linearly, e.g. $g_2(\zeta_4\, g_0 + g_4)g_8 = \zeta_4\, g_2g_0g_8 + g_2g_4g_8$. Nevertheless, the counting of indecomposable eMZVs remains unmodified when projecting to the maximal component of their γ-representation, see the discussion below eq. (4.17a).

5 Conclusions

In this work we have been studying the systematics of relations between eMZVs. Our results support the conjecture that the entirety of relations can be traced back to reflection, shuffle and Fay identities.

The numbers $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs at any weight and length can be explained once their connection to a special derivation algebra is taken into account: Any eMZV can be expressed in terms of iterated integrals over Eisenstein series whose appearance in turn is governed by the derivation algebra.

Our results for the numbers $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs for various weights w_ω and lengths ℓ_ω are listed in table 8. In addition, there are all-weight formulæ available for $\ell_\omega \leq 4$ and odd values of $w_\omega + \ell_\omega$,

$$N(2,w_\omega) = 1\ , \qquad N(3,w_\omega) = \left\lceil\frac{1}{6}w_\omega\right\rceil\ , \qquad N(4,w_\omega) = \left\lfloor\frac{1}{2} + \frac{1}{48}(w_\omega+5)^2\right\rfloor\ , \tag{5.1}$$

where the expression for $N(4, w_\omega)$ is conjectural. Because of the diversity of constraints originating from the derivation algebra as described in section 4, a closed formula for all weights and lengths is challenging to find and not yet available. A closely related issue is the computation of the dimensions of the Lie algebra $\mathfrak{u}$, which has been carried out by Brown for depths 1, 2 and 3 [43].

Explicit q-expansions for eMZVs are accessible using a slightly modified version of iterated Eisenstein integrals described in subsection 4.6. The resulting closed expression can be found in eq. (4.64).

The improved understanding of eMZVs raises a variety of follow-up questions, starting with a connection of the underlying elliptic iterated integrals with recent results on Feynman integrals [6–9]. In particular, the techniques which led to the q-expansions of eMZVs furnish a convenient starting point to connect with the functions ELi introduced in ref. [7] and generalized in ref. [9].

The appearance of eMZVs in one-loop scattering amplitudes of the open superstring [5] suggested a systematic study of indecomposable eMZVs. The results of the current article should pave the way towards a compact form of string corrections at higher orders in α' and might even lead to a glimpse of an all-order pattern. The existence of such a description is not unlikely: for open-string tree-level amplitudes a recursive formula based on the Drinfeld associator is known. It was found by extending an initial observation in ref. [44] into a recursive computation of the complete α'-expansions in ref. [45]. Similarly, the α'-expansion at one-loop might be accessible by using the elliptic associators discussed in ref. [16].

The α'-expansion of the closed-string four-point amplitude at genus one has been investigated in refs. [46–48], see [49, 50] for generalizations to five external states. The functions appearing in those amplitudes include non-holomorphic Eisenstein series and a variety of their generalizations which have been analyzed in ref. [48]. It would be interesting to establish a connection between these non-holomorphic functions and modular-invariant combinations of eMZVs and their counterpart originating from the other homology cycle.

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Appendix

A eMZV relations

A.1 Decomposition of boring eMZVs

By eq. (2.17), all the above examples of shuffle-reductions of boring eMZVs can be identified as special cases of the following general identity

$$\omega(B)\Big|_{\text{boring}} = \sum_{k=1}^{\infty} D_{2k} \sum_{\substack{B=A_1A_2\ldots A_{2k} \\ \omega(A_i) \ \text{interesting}}} \omega(A_1)\,\omega(A_2)\ldots\omega(A_{2k}) \ , \tag{A.1}$$

whose rational coefficients $D_2 = \frac{1}{2},\ D_4 = -\frac{1}{8}, \ldots$ are given by

$$D_{2k} = (-1)^{k-1} \frac{(2k-3)!!}{k!\, 2^k} \ . \tag{A.2}$$

The arguments $B \equiv n_1, n_2, \ldots, n_r$ of the boring eMZVs on the left-hand side are deconcatenated[7] into smaller tuples $A_j = a_1^j, a_2^j, \ldots, a_{m_j}^j$ such that all eMZVs $\omega(A_j)$ are interesting. Only even numbers of interesting $\omega(A_j)$ are compatible with the boring nature of $\omega(B)$, and the concatenation $A_j A_{j+1}$ in eq. (A.1) is defined to yield $a_1^j, \ldots, a_{m_j}^j, a_1^{j+1}, \ldots, a_{m_{j+1}}^{j+1}$.

Note that the first appearance of $D_4 = -\frac{1}{8}$ can be seen from the second case $\omega(n_1, n_2, n_3, n_4)$ in eq. (2.20). The vanishing of eMZVs with all entries odd (cf. eq. (2.21)) follows from the absence of deconcatenations into tuples A_j with $\omega(A_j)$ interesting.

In order to prove[8] eqs. (A.1) and (A.2), we recall that the antipode

$$\mathcal{S}(n_1, n_2, \ldots, n_r) \equiv (-1)^r (n_r, \ldots, n_2, n_1) \tag{A.3}$$

in the shuffle algebra of words $B = n_1, n_2, \ldots, n_r$ satisfies the following defining property [51]

$$B + \mathcal{S}(B) + \sum_{\substack{B=A_1A_2 \\ A_1, A_2 \neq \emptyset}} A_1 ш \mathcal{S}(A_2) = 0 \ , \qquad B \neq \emptyset \ . \tag{A.4}$$

Since boring and interesting eMZVs can be neatly characterized through the antipode eq. (A.3),

$$\omega(\mathcal{S}(B)) = \begin{cases} \ \ \omega(B) & : \ \omega(B) \ \text{boring} \\ -\omega(B) & : \ \omega(B) \ \text{interesting} \end{cases} \ , \tag{A.5}$$

applying $\omega(\cdot)$ to (A.4) with boring $\omega(B)$ yields

$$\omega(B)\Big|_{\text{boring}} = \frac{1}{2} \Big\{ \sum_{\substack{B=A_1A_2 \\ \omega(A_i) \ \text{interesting}}} \omega(A_1)\,\omega(A_2) - \sum_{\substack{B=B_1B_2 \\ \omega(B_i) \ \text{boring}}} \omega(B_1)\,\omega(B_2) \Big\} \ . \tag{A.6}$$

This formula can be recursively applied to the boring factors $\omega(B_i)$ on the right-hand side until only interesting contributions remain, leading to the structure of eq. (A.1). The coefficients D_{2k}

[7]For example, the $k = 1$ part of eq. (A.1) encompasses those deconcatenations $B = A_1 A_2$ into $A_1 = n_1, n_2, \ldots, n_j$ and $A_2 = n_{j+1}, \ldots, n_r$ where $\omega(n_1, n_2, \ldots, n_j)$ and $\omega(n_{j+1}, \ldots, n_r)$ are interesting eMZVs.

[8]We are grateful to an anonymous referee for suggesting the proof.

in eq. (A.2) are determined by the combinatorics of iterating eq. (A.6), e.g.

$$\omega(B)\Big|_{\text{boring}} = \frac{1}{2} \sum_{\substack{B=A_1A_2 \\ \omega(A_i)\ \text{interesting}}} \omega(A_1)\,\omega(A_2) - \frac{1}{8} \sum_{\substack{B=B_1B_2 \\ \omega(B_i)\ \text{boring}}} \Bigg\{ \sum_{\substack{B_1=A_1A_2 \\ \omega(A_i)\ \text{interesting}}} \omega(A_1)\,\omega(A_2) - \sum_{\substack{B_1=B_3B_4 \\ \omega(B_i)\ \text{boring}}} \omega(B_3)\,\omega(B_4) \Bigg\}$$
$$\times \Bigg\{ \sum_{\substack{B_2=A_3A_4 \\ \omega(A_i)\ \text{interesting}}} \omega(A_3)\,\omega(A_4) - \sum_{\substack{B_2=B_5B_6 \\ \omega(B_i)\ \text{boring}}} \omega(B_5)\,\omega(B_6) \Bigg\} \tag{A.7}$$

reproduces the first two terms with $k \le 2$ in eq. (A.1). In the next iteration step towards $k = 3$, either the first boring pair $\omega(B_3)\,\omega(B_4)$ or the second one $\omega(B_5)\,\omega(B_6)$ can be rearranged via eq. (A.6). Since both of them contribute $\frac{1}{32}\sum_{B=A_1\ldots A_6} \omega(A_1)\ldots\omega(A_6)$ with $\omega(A_i)$ interesting, the above coefficient $D_6 = \frac{1}{16}$ rests on the two subdivisions of the schematic form $\{(A_1A_2)(A_3A_4)\}(A_5A_6)$ or $(A_1A_2)\{(A_3A_4)(A_5A_6)\}$, referring to the application of eq. (A.6) to either $\omega(B_3)\,\omega(B_4)$ or $\omega(B_5)\,\omega(B_6)$, respectively.

The contributions of $\sum_{B=A_1\ldots A_{2k}}$ from the iteration of eq. (A.6) can be organized in terms of full binary trees with $k-1$ internal vertices and k leaves. Internal vertices represent the expansion of pairs of boring eMZVs via eq. (A.6) and at each of the k leaves, a pair of interesting eMZVs is kept. Hence, the coefficient of $\sum_{B=A_1\ldots A_{2k}} \omega(A_1)\ldots\omega(A_{2k})$ in eq. (A.1) is given by

$$D_{2k} = \frac{1}{2}\left(-\frac{1}{4}\right)^{k-1} \cdot \#(\text{full binary trees with } k \text{ leaves})\ , \tag{A.8}$$

where each additional application of eq. (A.6) to a pair of boring $\omega(B_i)$ involves a prefactor of $-\frac{1}{4}$. Finally, since full binary trees with k leaves are counted by the Catalan number C_{k-1} [52] with

$$C_n = \frac{(2n)!}{(n+1)!\,n!} = 2^n \frac{(2n-1)!!}{(n+1)!}\,, \tag{A.9}$$

we recover the coefficients $D_{2k} = \frac{(-1)^{k-1}}{2^{2k-1}}\,C_{k-1}$ in eq. (A.2) from eq. (A.8).

A.2 More general Fay identities

The relation eq. (2.25) among elliptic iterated integrals yields various Fay identities in the limit $z \to 1$ and generalizes as follows to multiple appearances of the argument among the labels:

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_k & n_{k+1} & \ldots & n_r \\ z & z & \ldots & z & 0 & \ldots & 0 \end{smallmatrix}; z\right) = (-1)^k\,\zeta^{ш}(\underbrace{0\ldots0}_{r-k}\underbrace{1\ldots1}_{k}) \prod_{j=1}^{r} \delta_{n_j,1}$$
$$- (-1)^{n_k} \int_0^z \mathrm{d}t\, f^{(n_k+n_{k+1})}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{k-1} & 0 & n_{k+2} & \ldots & n_r \\ t & \ldots & t & 0 & 0 & \ldots & 0 \end{smallmatrix}; t\right) \tag{A.10}$$
$$+ \sum_{j=0}^{n_{k+1}} \binom{n_k-1+j}{j} \int_0^z \mathrm{d}t\, f^{(n_{k+1}-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{k-1} & n_k+j & n_{k+2} & \ldots & n_r \\ t & \ldots & t & t & 0 & \ldots & 0 \end{smallmatrix}; t\right)$$
$$+ \sum_{j=0}^{n_k} \binom{n_{k+1}-1+j}{j} (-1)^{n_k+j} \int_0^z \mathrm{d}t\, f^{(n_k-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{k-1} & n_{k+1}+j & n_{k+2} & \ldots & n_r \\ t & \ldots & t & 0 & 0 & \ldots & 0 \end{smallmatrix}; t\right)\,.$$

The MZV in the first line stems from the limit $z \to 0$ of $\Gamma(\underbrace{\begin{smallmatrix} 1 & \cdots & 1 \\ z & \cdots & z \end{smallmatrix}}_{k}\underbrace{\begin{smallmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{smallmatrix}}_{r-k}; z)$, where every $f^{(1)}(z)$ can be replaced by $\frac{1}{z}$ in this regime. As explained in ref. [5], the elliptic iterated integrals then reduce to particular instances of multiple polylogarithms, which can be shown to yield MZVs in this case.

The first novel eMZV relations follow from the limit $z \to 1$ of eq. (A.10) at $k = 2$ and $r = 4, 5$:

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_1 & n_2 & n_3 & n_4 \\ z & z & 0 & 0 \end{smallmatrix}; z\right) &= -\frac{1}{4}\zeta_4\,\delta_{n_1,1}\delta_{n_2,1}\delta_{n_3,1}\delta_{n_4,1} - (-1)^{n_2}\int_0^z \mathrm{d}t\, f^{(n_2+n_3)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & 0 & n_4 \\ t & 0 & 0 \end{smallmatrix}; t\right) \\
&+ \sum_{j=0}^{n_3}\binom{n_2-1+j}{j}\int_0^z \mathrm{d}t\, f^{(n_3-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_2+j & n_4 \\ t & t & 0 \end{smallmatrix}; t\right) \\
&+ \sum_{j=0}^{n_2}\binom{n_3-1+j}{j}(-1)^{n_2+j}\int_0^z \mathrm{d}t\, f^{(n_2-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_3+j & n_4 \\ t & 0 & 0 \end{smallmatrix}; t\right)
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_1 & n_2 & n_3 & n_4 & n_5 \\ z & z & 0 & 0 & 0 \end{smallmatrix}; z\right) &= (2\,\zeta_5 - \zeta_2\,\zeta_3)\Big(\prod_{j=1}^{5}\delta_{n_j,1}\Big) - (-1)^{n_2}\int_0^z \mathrm{d}t\, f^{(n_2+n_3)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & 0 & n_4 & n_5 \\ t & 0 & 0 & 0 \end{smallmatrix}; t\right) \\
&+ \sum_{j=0}^{n_3}\binom{n_2-1+j}{j}\int_0^z \mathrm{d}t\, f^{(n_3-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_2+j & n_4 & n_5 \\ t & t & 0 & 0 \end{smallmatrix}; t\right) \\
&+ \sum_{j=0}^{n_2}\binom{n_3-1+j}{j}(-1)^{n_2+j}\int_0^z \mathrm{d}t\, f^{(n_2-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_3+j & n_4 & n_5 \\ t & 0 & 0 & 0 \end{smallmatrix}; t\right) .
\end{aligned} \tag{A.12}$$

In particular, note that the product $\zeta_2\,\zeta_3$ is absent in eq. (2.25) at $r = 5$. Also, note that the divergent nature of $f^{(1)}$ causes extra complications in the limit $z \to 1$ of eq. (A.11) if $n_i = 1$ for $i = 1, 2, 3, 4$ and eq. (A.12) if $n_2 = n_3 = n_4 = 1$ and one of $n_1 = 1$ or $n_5 = 1$.

B Iterated Eisenstein integrals versus eMZVs: examples

In this appendix, we supplement further examples for the conversion of eMZVs into modified iterated Eisenstein integrals as defined in eq. (4.54).

B.1 Conversion of $\omega(0, 0, \ldots, 0, n)$

For eMZVs with only one non-zero entry, a closed formula can be given for their conversion into iterated Eisenstein integrals. At length $\ell_\omega = 4$ and $\ell_\omega = 5$, eqs. (4.58a) and (4.58b) can be generalized to

$$\omega(0,0,0,n) = \delta_{n,1}\left(\frac{i\pi}{12} + \frac{\zeta_3}{4\pi^2}\right) + \frac{n}{3!}\,\gamma_0(n+1) + n(n+1)(n+2)\,\gamma_0(n+3,0,0) \tag{B.1}$$

$$\omega(0,0,0,0,n) = -\frac{2\,\zeta_n}{5!} - \frac{n}{3!}(n+1)\,\gamma_0(n+2,0) - n(n+1)(n+2)(n+3)\,\gamma_0(n+4,0,0,0)\ , \tag{B.2}$$

where n is chosen to be odd in eq. (B.1) and even in eq. (B.2). At arbitrary length ℓ, we have

$$\omega(\underbrace{0,0,\ldots,0}_{\ell-1},n) = \begin{cases} \omega_0(0^{\ell-1},n) + \displaystyle\sum_{\substack{i=1,3,5,\\ \ldots,\ell-1}} \frac{\gamma_0(n+i,0^{i-1})}{(\ell-i)!}\prod_{j=0}^{i-1}(n+j) & : \ \ell \text{ even, } n \text{ odd} \\ -\dfrac{2\,\zeta_n}{\ell!} - \displaystyle\sum_{\substack{i=2,4,6,\\ \ldots,\ell-1}} \frac{\gamma_0(n+i,0^{i-1})}{(\ell-i)!}\prod_{j=0}^{i-1}(n+j) & : \ \ell \text{ odd, } n \text{ even} \end{cases}\ , \tag{B.3}$$

where the constant term for odd values of n vanishes except for weight one,

$$\omega_0(0^{\ell-1},n) = \delta_{n,1}\left\{\frac{i\pi}{2(\ell-1)!} - \sum_{k=1}^{\lfloor \ell/2\rfloor -1}\frac{(-1)^k}{(\ell-(2k+1))!}\frac{\zeta_{2k+1}}{(4\pi^2)^k}\right\} . \tag{B.4}$$

The q-expansion of eq. (B.3) can be inferred from the special case of eq. (4.64),

$$\gamma_0(k,0^{p-1}) = -\frac{2(2\pi i)^{k-2p}}{(k-1)!}\sum_{n=1}^{\infty}\frac{\sigma_{k-1}(n)q^n}{n^p} , \tag{B.5}$$

see eq. (4.63) for the definition of the divisor sum $\sigma_k(n)$.

B.2 Conversion of indecomposable eMZVs at $\ell_\omega \geq 3$

Among the indecomposable eMZVs beyond $\omega(0,\ldots,0,n)$, the simplest case $\omega(0,3,5)$ is converted to (modified) iterated Eisenstein integrals in eqs. (4.15) and (4.61). Beyond that, we find for example

$$\begin{aligned}
\omega(0,3,7) &= -1848\,\gamma(12,0) - 294\,\gamma(8,4) + \text{nmt} \\
&= -1848\,\gamma_0(12,0) - 294\,\gamma_0(8,4) - 75\big(\gamma_0(6)\big)^2 + 588\,\zeta_4\,\gamma_0(8,0) - 504\,\zeta_8\,\gamma_0(4,0) \\
\omega(0,3,9) &= -5616\,\gamma(14,0) - 729\,\gamma(10,4) - 315\,\gamma(8,6) + \text{nmt} \\
&= -5616\,\gamma_0(14,0) - 729\,\gamma_0(10,4) - 315\,\gamma_0(8,6) - 210\,\gamma_0(6)\,\gamma_0(8) \\
&\quad + 1458\,\zeta_4\,\gamma_0(10,0) + 630\,\zeta_6\,\gamma_0(8,0) - 630\,\zeta_6\,\gamma_0(6,0) - 1350\,\zeta_{10}\,\gamma_0(4,0) \\
\omega(0,3,11) &= -13695\,\gamma(16,0) - 1452\,\gamma(12,4) - 990\,\gamma(10,6) + \text{nmt} \\
&= -13695\,\gamma_0(16,0) - 1452\,\gamma_0(12,4) - \frac{735}{2}\big(\gamma_0(8)\big)^2 - 990\,\gamma_0(10,6) - 270\,\gamma_0(6)\,\gamma_0(10) \\
&\quad + 2904\,\zeta_4\,\gamma_0(12,0) + 1980\,\zeta_6\,\gamma_0(10,0) - 1980\,\zeta_{10}\,\gamma_0(6,0) - 2772\,\zeta_{12}\,\gamma_0(4,0) \\
\omega(0,5,9) &= -30105\,\gamma(16,0) - 5445\,\gamma(12,4) - 3105\,\gamma(10,6) + \text{nmt} \\
&= -30105\,\gamma_0(16,0) - 5445\,\gamma_0(12,4) - 3105\,\gamma_0(10,6) - \frac{735}{2}\big(\gamma_0(8)\big)^2 \\
&\quad + 10890\,\zeta_4\,\gamma_0(12,0) + 6210\,\zeta_6\,\gamma_0(10,0) - 5850\,\zeta_{10}\,\gamma_0(6,0) - 8910\,\zeta_{12}\,\gamma_0(4,0)
\end{aligned} \tag{B.6}$$

at length three, and

$$\begin{aligned}
\omega(0,0,2,3) &= 252\,\gamma(8,0,0) - 18\,\gamma(4,4,0) + \frac{5}{6}\,\gamma(6) + \text{nmt} \\
&= 252\,\gamma_0(8,0,0) - 18\,\gamma_0(4,4,0) + \frac{5}{6}\,\gamma_0(6) - 72\,\zeta_4\,\gamma_0(4,0,0) \\
\omega(0,0,2,5) &= 2826\,\gamma(10,0,0) + 150\,\gamma(6,4,0) + 180\,\gamma(6,0,4) + \frac{7}{6}\,\gamma(8) + \text{nmt} \\
&= 2826\,\gamma_0(10,0,0) + 150\,\gamma_0(6,4,0) + 180\,\gamma_0(6,0,4) + \frac{7}{6}\,\gamma_0(8) \\
&\quad - 660\,\zeta_4\,\gamma_0(6,0,0) + 180\,\zeta_6\,\gamma_0(4,0,0) \\
\omega(0,0,4,3) &= -2340\,\gamma(10,0,0) - 300\,\gamma(6,4,0) - 120\,\gamma(6,0,4) + \frac{7}{6}\,\gamma(8) + \text{nmt} \\
&= -2340\,\gamma_0(10,0,0) - 300\,\gamma_0(6,4,0) - 120\,\gamma_0(6,0,4) - 60\,\gamma_0(4)\,\gamma_0(6,0) + \frac{7}{6}\,\gamma_0(8) \\
&\quad + 480\,\zeta_4\,\gamma_0(6,0,0) - 1080\,\zeta_6\,\gamma_0(4,0,0) - 3\,\zeta_4\,\gamma_0(4)
\end{aligned} \tag{B.7}$$

at length four, where "nmt" refers to non-maximal terms as explained after eq. (4.17a). The q-expansion of the constituents is given by eq. (4.64).

C Examples for relations in the derivation algebra $\mathfrak{u}$

C.1 Known relations

Irreducible relations $r^{\text{depth}}_{\text{weight}}$ are listed in table 5. For depth two, all relations can be obtained from eq. (4.42). At depth three, we can confirm the relations listed in eq. (4.28c) as well as [24]:

$$\begin{aligned} r^3_{20}: \quad 0 =\ & 1050[\epsilon_0,[\epsilon_6,\epsilon_{14}]] - 6580[\epsilon_0,[\epsilon_8,\epsilon_{12}]] + 4320[\epsilon_4,[\epsilon_0,\epsilon_{16}]] - 10970[\epsilon_4,[\epsilon_4,\epsilon_{12}]] \\ & + 166675[\epsilon_4,[\epsilon_6,\epsilon_{10}]] - 17150[\epsilon_6,[\epsilon_0,\epsilon_{14}]] - 500675[\epsilon_6,[\epsilon_6,\epsilon_8]] + 30184[\epsilon_8,[\epsilon_0,\epsilon_{12}]] \\ & + 80388[\epsilon_8,[\epsilon_4,\epsilon_8]] - 17325[\epsilon_{10},[\epsilon_0,\epsilon_{10}]] \end{aligned} \tag{C.1}$$

$$\begin{aligned} r^3_{22}: \quad 0 =\ & 40[\epsilon_0,[\epsilon_6,\epsilon_{16}]] - 280[\epsilon_0,[\epsilon_8,\epsilon_{14}]] + 910[\epsilon_0,[\epsilon_{10},\epsilon_{12}]] - 360[\epsilon_4,[\epsilon_0,\epsilon_{18}]] \\ & - 11535[\epsilon_4,[\epsilon_6,\epsilon_{12}]] + 6069[\epsilon_4,[\epsilon_8,\epsilon_{10}]] + 1320[\epsilon_6,[\epsilon_0,\epsilon_{16}]] + 15140[\epsilon_6,[\epsilon_4,\epsilon_{12}]] \\ & - 7150[\epsilon_6,[\epsilon_6,\epsilon_{10}]] - 1820[\epsilon_8,[\epsilon_0,\epsilon_{14}]] - 12922[\epsilon_8,[\epsilon_6,\epsilon_8]] + 858[\epsilon_{10},[\epsilon_0,\epsilon_{12}]] \end{aligned} \tag{C.2}$$

$$\begin{aligned} r^4_{18}: \quad 0 =\ & [\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_{12}]]] - \frac{215}{74}[\epsilon_0,[\epsilon_0,[\epsilon_8,\epsilon_{10}]]] - \frac{2323}{518}[\epsilon_0,[\epsilon_4,[\epsilon_6,\epsilon_8]]] + \frac{218}{37}[\epsilon_0,[\epsilon_6,[\epsilon_4,\epsilon_8]]] \\ & + \frac{60}{407}[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]] + \frac{285561}{5698}[\epsilon_4,[\epsilon_0,[\epsilon_6,\epsilon_8]]] + \frac{8599}{1628}[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_{10}]]] \\ & + \frac{53855}{444}[\epsilon_4,[\epsilon_4,[\epsilon_4,\epsilon_6]]] - \frac{691}{333}[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]] - \frac{19853}{518}[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_8]]] \\ & - \frac{691}{74}[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_8]]] + \frac{691}{111}[\epsilon_{12},[\epsilon_0,[\epsilon_0,\epsilon_6]]] - \frac{60}{37}[\epsilon_{14},[\epsilon_0,[\epsilon_0,\epsilon_4]]] - \frac{87595}{1554}[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_8]]] \\ & + \frac{17275}{333}[\epsilon_6,[\epsilon_6,[\epsilon_0,\epsilon_6]]] + \frac{3455}{518}[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]] + \frac{49565}{518}[\epsilon_8,[\epsilon_0,[\epsilon_4,\epsilon_6]]] \end{aligned} \tag{C.3}$$

$$\begin{aligned} r^4_{22}: \quad 0 =\ & [\epsilon_0,[\epsilon_0,[\epsilon_8,\epsilon_{14}]]] + \frac{192903}{230}[\epsilon_0,[\epsilon_4,[\epsilon_6,\epsilon_{12}]]] - \frac{861492}{805}[\epsilon_0,[\epsilon_6,[\epsilon_4,\epsilon_{12}]]] \\ & + \frac{134488}{161}[\epsilon_0,[\epsilon_6,[\epsilon_6,\epsilon_{10}]]] + \frac{6588}{805}[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{18}]]] + \frac{269217}{805}[\epsilon_4,[\epsilon_0,[\epsilon_6,\epsilon_{12}]]] \\ & - \frac{39418}{115}[\epsilon_4,[\epsilon_0,[\epsilon_8,\epsilon_{10}]]] - \frac{13253}{115}[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_{14}]]] - \frac{18221}{115}[\epsilon_4,[\epsilon_4,[\epsilon_6,\epsilon_8]]] \\ & + \frac{33109}{322}[\epsilon_6,[\epsilon_0,[\epsilon_6,\epsilon_{10}]]] + \frac{25095129}{37375}[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_{12}]]] + \frac{11266827}{5750}[\epsilon_6,[\epsilon_4,[\epsilon_4,\epsilon_8]]] \\ & - \frac{786557}{644}[\epsilon_6,[\epsilon_6,[\epsilon_0,\epsilon_{10}]]] + \frac{80233}{1265}[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]] + \frac{21742068}{6325}[\epsilon_8,[\epsilon_0,[\epsilon_6,\epsilon_8]]] \\ & - \frac{112835}{253}[\epsilon_8,[\epsilon_4,[\epsilon_0,\epsilon_{10}]]] + \frac{403764}{115}[\epsilon_8,[\epsilon_4,[\epsilon_4,\epsilon_6]]] + \frac{644938}{575}[\epsilon_8,[\epsilon_6,[\epsilon_0,\epsilon_8]]] \\ & - \frac{103859}{115}[\epsilon_{10},[\epsilon_0,[\epsilon_4,\epsilon_8]]] + \frac{301851}{8050}[\epsilon_{12},[\epsilon_0,[\epsilon_0,\epsilon_{10}]]] + \frac{734133}{805}[\epsilon_{12},[\epsilon_0,[\epsilon_4,\epsilon_6]]] \\ & - \frac{493889}{8050}[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_{12}]]] - \frac{372888}{10465}[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{16}]]] - \frac{23054063}{52325}[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_{12}]]] \\ & - \frac{1015637}{1150}[\epsilon_0,[\epsilon_4,[\epsilon_8,\epsilon_{10}]]] - \frac{27458211}{3220}[\epsilon_6,[\epsilon_6,[\epsilon_4,\epsilon_6]]] - \frac{23679}{8050}[\epsilon_0,[\epsilon_0,[\epsilon_{10},\epsilon_{12}]]] \\ & - \frac{1913}{115}[\epsilon_{14},[\epsilon_0,[\epsilon_0,\epsilon_8]]] + \frac{672}{115}[\epsilon_{16},[\epsilon_0,[\epsilon_0,\epsilon_6]]] - \frac{972}{805}[\epsilon_{18},[\epsilon_0,[\epsilon_0,\epsilon_4]]] \end{aligned} \tag{C.4}$$

C.2 New relations

At depth 5 we explicitly isolated the irreducible relation r_{20}^5, which is apparently new:

$$
\begin{aligned}
r_{20}^5: \quad 0 = {} & 2206388620800\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_{16}]]]] - 8366188740000\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_{14}]]]] \\
& + 12305858292000\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_8,\epsilon_{12}]]]] - 1834700544000\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{16}]]]] \\
& + 35687825530800\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_4,\epsilon_{12}]]]] - 109425220173750\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_6,\epsilon_{10}]]]] \\
& - 39970750599360\,[\epsilon_0,[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_{12}]]]] - 380488416808500\,[\epsilon_0,[\epsilon_4,[\epsilon_4,[\epsilon_4,\epsilon_8]]]] \\
& + 13171256280000\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]]] + 220479512028750\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_{10}]]]] \\
& - 498847136287500\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_6,\epsilon_8]]]] + 220479512028750\,[\epsilon_0,[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_{10}]]]] \\
& - 458212979593200\,[\epsilon_0,[\epsilon_6,[\epsilon_4,[\epsilon_4,\epsilon_6]]]] + 17540335312500\,[\epsilon_0,[\epsilon_6,[\epsilon_6,[\epsilon_0,\epsilon_8]]]] \\
& - 34407225652800\,[\epsilon_0,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]] - 97419791414400\,[\epsilon_0,[\epsilon_8,[\epsilon_0,[\epsilon_4,\epsilon_8]]]] \\
& - 197536749664800\,[\epsilon_0,[\epsilon_8,[\epsilon_4,[\epsilon_0,\epsilon_8]]]] + 22970739577500\,[\epsilon_0,[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]] \\
& + 161385266688750\,[\epsilon_0,[\epsilon_{10},[\epsilon_0,[\epsilon_4,\epsilon_6]]]] + 611566848000\,[\epsilon_4,[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{16}]]]] \\
& - 58836403790864\,[\epsilon_4,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_{12}]]]] + 134572047805000\,[\epsilon_4,[\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_{10}]]]] \\
& + 965866444426884\,[\epsilon_4,[\epsilon_0,[\epsilon_4,[\epsilon_4,\epsilon_8]]]] + 92810063342256\,[\epsilon_4,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]] \\
& - 204658497503460\,[\epsilon_4,[\epsilon_4,[\epsilon_0,[\epsilon_4,\epsilon_8]]]] + 541534390897500\,[\epsilon_4,[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_8]]]] \\
& - 215755493216250\,[\epsilon_4,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]] - 1490371718737200\,[\epsilon_4,[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_6]]]] \\
& + 1032598095322950\,[\epsilon_4,[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_6]]]] + 298655975581600\,[\epsilon_4,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_8]]]] \\
& - 220479512028750\,[\epsilon_4,[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_6]]]] + 54837332264496\,[\epsilon_4,[\epsilon_{12},[\epsilon_0,[\epsilon_0,\epsilon_4]]]] \\
& - 6941740260000\,[\epsilon_6,[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]]] - 220479512028750\,[\epsilon_6,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_{10}]]]] \\
& + 231883232831250\,[\epsilon_6,[\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_8]]]] + 519528504682200\,[\epsilon_6,[\epsilon_0,[\epsilon_4,[\epsilon_4,\epsilon_6]]]] \\
& - 220479512028750\,[\epsilon_6,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]] - 2120947122294000\,[\epsilon_6,[\epsilon_4,[\epsilon_0,[\epsilon_4,\epsilon_6]]]] \\
& - 1538522546497950\,[\epsilon_6,[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_6]]]] + 249423568143750\,[\epsilon_6,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_8]]]] \\
& - 266963903456250\,[\epsilon_6,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_6]]]] + 23162632092600\,[\epsilon_8,[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]] \\
& + 184988881773150\,[\epsilon_8,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_8]]]] + 310347440367510\,[\epsilon_8,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_8]]]] \\
& - 183822714075000\,[\epsilon_8,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_6]]]] + 171943360038450\,[\epsilon_8,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_4]]]] \\
& - 22551859687500\,[\epsilon_{10},[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]] - 240755752121625\,[\epsilon_{10},[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_6]]]] \\
& - 104628710038125\,[\epsilon_{10},[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_6]]]] - 14987648446875\,[\epsilon_{10},[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_4]]]] \\
& + 11918038532400\,[\epsilon_{12},[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_8]]]] + 46293152724000\,[\epsilon_{12},[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_4]]]] \\
& - 8366188740000\,[\epsilon_{14},[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_6]]]]\,. \qquad \text{(C.5)}
\end{aligned}
$$

The complete set of all irreducible relations known to us is available from

`https://tools.aei.mpg.de/emzv` ,

and all of them have been verified by evaluating the action on the letters x, y via eq. (4.26).

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Appendix E

The meta-abelian elliptic KZB associator and periods of Eisenstein series

The meta-abelian elliptic KZB associator and periods of Eisenstein series

Nils Matthes

Abstract

We consider the image of Enriquez's elliptic KZB associator in the meta-abelian quotient of the fundamental Lie algebra of a once-punctured elliptic curve. Our main result is an explicit formula for this image, which involves Zagier's extended period polynomials of Eisenstein series, as well as a certain subset of the iterated Eisenstein integrals, introduced by Manin and Brown.

1 Introduction

Let τ be a point in the upper half-plane, and let $E_\tau^\times := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)\setminus\{0\}$ be the associated once-punctured, complex elliptic curve. The *elliptic Knizhnik-Zamolodchikov-Bernard (KZB) equation* [2, 10, 6] is the universal differential equation on $E_\tau^\times$ with unipotent monodromy around 0. It plays an important role in a variety of subjects, such as elliptic braid groups [2], invariants of 3-manifolds [8], and universal mixed elliptic motives [7]. The monodromy of the elliptic KZB equation is described by the *elliptic KZB associator* [5]

$$(\Phi, \underline{A}(\tau), \underline{B}(\tau)) \in \mathbb{C}\langle\langle x,y\rangle\rangle \times \mathbb{C}\langle\langle x,y\rangle\rangle \times \mathbb{C}\langle\langle x,y\rangle\rangle, \tag{1.1}$$

where $\mathbb{C}\langle\langle x,y\rangle\rangle$ denotes the $\mathbb{C}$-algebra of formal power series in non-commuting variables x and y. Here Φ is the Drinfeld associator [3] and $\underline{A}(\tau)$ and $\underline{B}(\tau)$ are holomorphic functions of τ. The elliptic KZB associator satisfies several functional equations reflecting its relation to elliptic braid groups, as well as a differential equation, which is closely connected to a certain Lie algebra of derivations [18, 15].

Additional structure of the elliptic KZB associator is revealed by considering its formal logarithm $(\varphi, \underline{\mathfrak{A}}(\tau), \underline{\mathfrak{B}}(\tau)) \in \widehat{\mathcal{L}} \times \widehat{\mathcal{L}} \times \widehat{\mathcal{L}}$. Here, $\widehat{\mathcal{L}}$ is the completion (for the lower central series) of the free Lie algebra on the set $\{x,y\}$. Let $\widehat{\mathcal{L}}^{(1)} := [\widehat{\mathcal{L}}, \widehat{\mathcal{L}}]$ be the commutator, and $\widehat{\mathcal{L}}^{(2)} := [\widehat{\mathcal{L}}^{(1)}, \widehat{\mathcal{L}}^{(1)}]$ the double commutator of $\widehat{\mathcal{L}}$. Write

$$\widehat{\mathcal{L}}^{\mathrm{met-ab}} := \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)} \tag{1.2}$$

for the *meta-abelian quotient* of $\widehat{\mathcal{L}}$. Given $f \in \widehat{\mathcal{L}}$, we will denote by $f^{\mathrm{met-ab}}$ its image in $\widehat{\mathcal{L}}^{\mathrm{met-ab}}$. It is well-known that there is a natural isomorphism $\widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)} \cong \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(1)} \oplus \widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)} \cong (\mathbb{C}x \oplus \mathbb{C}y) \oplus \mathbb{C}[[X,Y]]$, and we will usually identify elements of $\widehat{\mathcal{L}}^{\mathrm{met-ab}}$ under this isomorphism, i.e. we write $f^{\mathrm{met-ab}} = f^{(0)} + f^{(1)}$, where $f^{(0)} \in \mathbb{C}x \oplus \mathbb{C}y$ and $f^{(1)} \in \mathbb{C}[[X,Y]]$. In this paper, we are mainly interested in the image

$$(\varphi^{\mathrm{met-ab}}, \underline{\mathfrak{A}}(\tau)^{\mathrm{met-ab}}, \underline{\mathfrak{B}}(\tau)^{\mathrm{met-ab}}) \in \widehat{\mathcal{L}}^{\mathrm{met-ab}} \times \widehat{\mathcal{L}}^{\mathrm{met-ab}} \times \widehat{\mathcal{L}}^{\mathrm{met-ab}}, \tag{1.3}$$

of the logarithm of the elliptic KZB associator in the meta-abelian quotient of $\widehat{\mathcal{L}}$. To begin with, the computation of $\varphi^{\mathrm{met-ab}}$ is a classical result of Drinfeld.

Proposition 1.1 ([3], §2)**.** *Let $\varphi^{\mathrm{met-ab}} = \varphi^{(0)} + \varphi^{(1)}$ as above. Then $\varphi^{(0)} = 0$ and*

$$\begin{aligned}\varphi^{(1)} &= (X\cdot Y)^{-1}\left[\exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n\cdot(2\pi i)^n}(X^n+Y^n-(X+Y)^n)\right)-1\right]\\ &= (\overline{X}\cdot\overline{Y})^{-1}\left[\frac{\Gamma(1-\overline{X})\Gamma(1-\overline{Y})}{\Gamma(1-(\overline{X}+\overline{Y}))}-1\right],\end{aligned} \tag{1.4}$$

where $\overline{X} = X/2\pi i$ and $\overline{Y} = Y/2\pi i$. Here, $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\mathrm{d}t$ denotes the classical Gamma function, and the last line is defined by the expansion $\log(\Gamma(1-z)) = \gamma z + \sum_{k\geq 2}\frac{\zeta(k)}{k}z^k$, where γ is Euler's constant.

The main goal of the present paper is to find a formula for $\underline{\mathfrak{A}}(\tau)^{\mathrm{met-ab}}$ and $\underline{\mathfrak{B}}(\tau)^{\mathrm{met-ab}}$, thus completing the description of (1.3). In order to simplify the resulting formulas, we will make the change of variables $a := 2\pi i x$ and $b := y$, as well as $A^k := (2\pi i)^{k+1}X^k$ and $B^k = 2\pi i Y^k$. It is relatively easy to show that (cf. Proposition 3.2)

$$\underline{\mathfrak{A}}(\tau)^{(0)} = -b, \quad \underline{\mathfrak{B}}(\tau)^{(0)} = a - \tau b. \tag{1.5}$$

In order to write down a formula for $\underline{\mathfrak{A}}(\tau)^{(1)}$ and $\underline{\mathfrak{B}}(\tau)^{(1)}$, we need *iterated Eisenstein integrals* [11, 1]

$$\mathcal{E}(2k_1, \ldots, 2k_n; \tau) := \int_\tau^{i\infty} E_{2k_1}(\tau_1)\mathrm{d}\tau_1 \ldots E_{2k_n}(\tau_n)\mathrm{d}\tau_n, \quad k_1, \ldots, k_n \geq 0. \tag{1.6}$$

Our conventions are the same as in [1], in particular the integration starts on the left, and $E_{2k}(\tau) := -\frac{B_{2k}}{4k} + \sum_{n\geq 1}\sigma_{2k-1}(n)q^n$ the Hecke-normalized Eisenstein series (we also set $E_0(\tau) = -1$). Here as usual, B_{2k} denotes the $2k$-th Bernoulli number, $\sigma_m(n) := \sum_{d|n} d^m$ is the m-th divisor function, and $q = e^{2\pi i \tau}$.

Theorem 1.2. *Let $\underline{\mathfrak{A}}_\infty^{(1)}$ be the value of $\underline{\mathfrak{A}}(\tau)^{(1)}$ at the tangential base point $\vec{1}_\infty$ at $i\infty$ [1], and define $\underline{\mathfrak{B}}_\infty^{(1)}$ likewise. We have:*

(i)

$$\underline{\mathfrak{A}}(\tau)^{(1)} = \underline{\mathfrak{A}}_\infty^{(1)} + \sum_{m\geq 0,\, n\geq 1} \frac{2}{(m+n-1)!}\alpha_{m,n}(\tau)\left(-B\frac{\partial}{\partial A}\right)^{n-1} A^{m+n-1}B, \tag{1.7}$$

where

$$\underline{\mathfrak{A}}_\infty^{(1)} = -\left(\sum_{k\geq 2}\lambda_k A^{k-1} + \frac{1}{4}B - \sum_{k\geq 3,\mathrm{odd}} \frac{\zeta(k)}{(2\pi i)^k}B^k\right) \tag{1.8}$$

$\lambda_k := \frac{B_k}{k!}$, and $\alpha_{m,n}(\tau) = -\mathcal{E}(\{0\}_{n-1}, m+n+1; \tau) + \frac{B_{m+n+1}}{2(m+n+1)}\mathcal{E}(\{0\}_n; \tau)$. In particular, $\alpha_{m,n}(\tau) = 0$, if $m+n \geq 1$ is even.

(ii)

$$\begin{aligned}\underline{\mathfrak{B}}(\tau)^{(1)} = \underline{\mathfrak{B}}_\infty^{(1)} &- \sum_{r\geq 1}\mathcal{E}(\{0\}_r;\tau)\sum_{m,n\geq 0} c_{m,n}\left[\left(-B\frac{\partial}{\partial A}\right)^r A^m B^n\right] \\ &+ \left[\sum_{k\geq 1}\frac{2}{(2k-2)!}\left\{\mathcal{E}(\{0\}_{r-1}, 2k;\tau) + \frac{1}{2k-1}\mathcal{E}(\{0\}_{r-2}, 2k, 0;\tau)\right\}\left(-B\frac{\partial}{\partial A}\right)^{r-1}A^{2k-1}\right],\end{aligned} \tag{1.9}$$

where

$$\underline{\mathfrak{B}}_\infty^{(1)} = -\left(\sum_{k\geq 2}\lambda_k B^{k-1} + \sum_{k\geq 3,\,\mathrm{odd}}\frac{\zeta(k)}{(2\pi i)^k}AB^{k-1} + \sum_{m,n\geq 2}\lambda_m\lambda_n A^m B^{n-1}\right). \tag{1.10}$$

Here, we set $\mathcal{E}(\{0\}_{-1}, 2k, 0; \tau) := 0$, and $c_{m,n}$ is defined as the coefficient of A^mB^n in $\underline{\mathfrak{B}}_\infty^{(1)}$.

This theorem can be seen as an analogue of a formula of Nakamura ([13], Theorem 3.3) for the Galois action on the meta-abelian quotient of the étale fundamental group of a once-punctured elliptic curve [13, 14]. Interestingly, the proof of Theorem 1.2 uses no linear or algebraic relation between iterated Eisenstein integrals. In fact, with the exception of the shuffle relations, there are no algebraic relations between iterated Eisenstein integrals [12].

We now describe a relation between the series $\underline{\mathfrak{A}}_\infty^{(1)}$, $\underline{\mathfrak{B}}_\infty^{(1)}$ and period polynomials. First, recall that one can associate to every cusp form f of weight k for $\mathrm{SL}_2(\mathbb{Z})$ a homogeneous polynomial $r_f(A,B) \in \mathbb{C}[A,B]$ of degree $k-2$, called its *period polynomial* (cf. e.g. [9]). In [20], Zagier has extended the notion of period polynomial to arbitrary modular forms (but still for $\mathrm{SL}_2(\mathbb{Z})$); in particular, he defines the *extended period polynomial* $r_{E_{2k}}(A,B)$ of the Eisenstein series E_{2k}. This is no longer a polynomial, but lives in the slightly bigger space $\bigoplus_{-1\leq n\leq 2k-1}\mathbb{C}\cdot A^nB^{2k-2-n}$.

Theorem 1.3. *The extended period polynomial $r_{E_{2k}}(A,B)$ of the Eisenstein series E_{2k} equals*

$$\frac{(2k-2)!}{2}\left(\widetilde{\mathfrak{A}}(A,B)^+_{2k-2}+\widetilde{\mathfrak{B}}(B,A)^+_{2k-2}+\widetilde{\mathfrak{A}}(A,B)^-_{2k-2}+\widetilde{\mathfrak{B}}(A,B)^-_{2k-2}\right), \tag{1.11}$$

where $\widetilde{\mathfrak{A}}(A,B)=B^{-1}\underline{\mathfrak{A}}^{(1)}_\infty(A,B)$ and $\widetilde{\mathfrak{B}}(A,B)=A^{-1}\underline{\mathfrak{B}}^{(1)}_\infty(A,B)$. Here, a subscript $2k-2$ denotes the homogeneous component of degree $2k-2$, and the superscript $+$ (resp. $-$) denotes the invariants (resp. anti-invariants) with respect to $(A,B)\mapsto(-A,B)$.

Put differently, the series $\underline{\mathfrak{A}}^{(1)}_\infty$ and $\underline{\mathfrak{B}}^{(1)}_\infty$ can be viewed as the generating series for the extended period polynomials of Eisenstein series.

We briefly describe the content of the paper. In Section 2, we collect some background on the elliptic KZB associator. We also recall the definition of a certain family of derivations on $\widehat{\mathcal{L}}$ (the *Eisenstein derivations*), which play a key role in the study of the elliptic KZB associator. In Section 3, we compute the action of these derivations on the meta-abelian quotient of $\widehat{\mathcal{L}}$. This computation is needed in Section 4, where we prove Theorems 1.2 and 1.3.

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2 Preliminaries

Let $\mathcal{L}$ denote the free $\mathbb{C}$-Lie algebra on the set $\{x,y\}$, and $\widehat{\mathcal{L}}$ its completion with respect to the lower central series. A good reference for free Lie algebras is [17].

2.1. Eisenstein derivations. We begin by describing a family of derivations on $\mathcal{L}$, which were first introduced in a slightly different context in [18]. For $k\geq 0$, define $\varepsilon_{2k}:\mathcal{L}\to\mathcal{L}$ by

$$\varepsilon_{2k}(x)=\mathrm{ad}^{2k}(x)(y),\quad \varepsilon_{2k}(y)=\sum_{0\leq j<k}(-1)^j[\mathrm{ad}^j(x)(y),\mathrm{ad}^{2k-1-j}(x)(y)], \tag{2.1}$$

where $\mathrm{ad}(x)(y):=[x,y]$ and $\mathrm{ad}^k(x)(y):=[x,\mathrm{ad}^{k-1}(x)(y)]$ for $k\geq 1$. Note that every ε_{2k} annihilates the commutator $[x,y]$:

$$\varepsilon_{2k}([x,y])=0,\quad \forall k\geq 0, \tag{2.2}$$

(cf. [15]). Since the derivations ε_{2k} preserve the lower central series of $\mathcal{L}$, they pass to $\widehat{\mathcal{L}}$. We also set

$$\widetilde{\varepsilon}_{2k}=\begin{cases}\frac{2}{(2k-2)!}\varepsilon_{2k} & k>0\\ -\varepsilon_0 & k=0,\end{cases} \tag{2.3}$$

and call these derivations *Eisenstein derivations.*[1]

2.2. The elliptic KZB associator. Consider the following classical Kronecker-Eisenstein series [19, 20]

$$F_\tau(\xi,\alpha):=\frac{\theta'_\tau(0)\theta_\tau(\xi+\alpha)}{\theta_\tau(\xi)\theta_\tau(\alpha)},\quad \xi,\alpha\in\mathbb{C} \tag{2.4}$$

with $\theta_\tau(\xi)$ the standard odd Jacobi theta function, and $\tau\in\mathbb{H}:=\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)>0\}$. As proved for example in [20], the Kronecker-Eisenstein series is meromorphic and has a simple pole at $\alpha=0$. In particular it possesses a Laurent expansion $F_\tau(\xi,\alpha)=\sum_{k\geq 0}f^{(k)}(\xi)\alpha^{k-1}$, where the $f^{(k)}(\xi)$ are meromorphic. We also set $\mathrm{ad}(x)F_\tau(\xi,\mathrm{ad}(x))(y)=\sum_{k\geq 0}f^{(k)}(\xi)\,\mathrm{ad}^k(x)(y)$.

[1] This name is justified by the fact that, for $k\geq 0$, the derivation $\widetilde{\varepsilon}_{2k}$ is the image of the Eisenstein generator $\mathbf{e}_{2k}\in\mathfrak{u}^{\mathrm{geom}}$ under the monodromy morphism $\mathfrak{u}^{\mathrm{geom}}\to\mathrm{Der}(\mathcal{L})$, where $\mathfrak{u}^{\mathrm{geom}}$ denotes the geometric fundamental Lie algebra of a certain category of mixed elliptic motives [7].

Now let $U := \{c + d\tau \,|\, c, d \in (0,1)\} \subset \mathbb{C} \setminus (\mathbb{Z} + \mathbb{Z}\tau)$. As shown in [5], there exist unique solutions $G_0, G_1 : U \to \mathbb{C}\langle\langle x, y\rangle\rangle$ to the equation

$$\frac{\partial g}{\partial \xi} = -\operatorname{ad}(x) F_\tau(\xi, \operatorname{ad}(x))(y) \cdot g \tag{2.5}$$

with the asymptotic $G_0(s) \sim (-2\pi i s)^{-[x,y]}$ as $s \to 0$ and $G_1(s) \sim (-2\pi i(1-s))^{-[x,y]}$ as $s \to 1$. Likewise, there exist unique solutions $H_0, H_1 : U \to \mathbb{C}\langle\langle x, y\rangle\rangle$ to the equation

$$\frac{\partial h}{\partial \xi} = \left(\frac{2\pi i}{\tau} \cdot x - \operatorname{ad}(x) e^{\frac{2\pi i}{\tau} \operatorname{ad}(x)} F_\tau(\xi, \operatorname{ad}(x))(y) \right) \cdot h \tag{2.6}$$

with the asymptotic $H_0(s) \sim (-2\pi i s)^{-[x,y]}$ as $s \to 0$ and $H_1(s) \sim (-2\pi i(1-s))^{-[x,y]}$.

Definition 2.1 (Enriquez, [5], §6.2)**.** The *elliptic KZB associator* is the triple $(\Phi, \underline{A}(\tau), \underline{B}(\tau)) \in (\mathbb{C}\langle\langle x, y\rangle\rangle)^3$, where Φ denotes the Drinfeld associator, and

$$\underline{A}(\tau) = G_1^{-1}(s) G_0(s), \quad \underline{B}(\tau) = H_1^{-1}(s) H_0(s). \tag{2.7}$$

By a standard result on linear differential equations (cf. [16], Theorem 3.2), there exist formal Lie series $\underline{\mathfrak{A}}(\tau), \underline{\mathfrak{B}}(\tau) \in \widehat{\mathcal{L}}$ such that

$$\underline{\mathfrak{A}}(\tau) = \log(\underline{A}(\tau)), \quad \underline{\mathfrak{B}}(\tau) = \log(\underline{B}(\tau)). \tag{2.8}$$

Remark 2.2. The original definition of the elliptic KZB associator given in [5] is slightly different, but essentially equivalent to the one given here. More precisely, denoting by $(\Phi, A(\tau), B(\tau))$ the version introduced in [5], we have $\underline{A}(\tau) = e^{-\pi i[x,y]} A(\tau)$ and $\underline{B}(\tau) = e^{\pi i[x,y]} B(\tau)$, i.e. the two definitions differ only by constant prefactors.

The elliptic KZB associator can be expressed using iterated Eisenstein integrals. To simplify the formulas, it is useful to work with the variables $a = 2\pi i x$ and $b = y$ instead of x, y. Let

$$g(\tau) = \sum \mathcal{E}(\underline{2k}; \tau) \widetilde{\varepsilon}_{\underline{2k}}, \tag{2.9}$$

where the sum is over all multi-indices $\underline{2k} := (2k_1, \ldots, 2k_n)$. Here, we set $\widetilde{\varepsilon}_{\underline{2k}} := \widetilde{\varepsilon}_{2k_1} \circ \ldots \circ \widetilde{\varepsilon}_{2k_n}$ and $\mathcal{E}(\underline{2k}; \tau)$ denotes the indefinite iterated Eisenstein integral, as in (1.6). Since $g(\tau)$ satisfies a linear differential equation with values in the Lie algebra of derivations $\operatorname{Der}(\widehat{\mathcal{L}})$, it follows that for fixed τ, $g(\tau)$ is a continuous automorphism of $\mathbb{C}\langle\langle a, b\rangle\rangle$.

Proposition 2.3 (Enriquez)**.** *We have $\underline{A}(\tau) = g(\tau)(\underline{A}_\infty)$ and $\underline{B}(\tau) = g(\tau)(\underline{B}_\infty)$, where*

$$\underline{A}_\infty = e^{\frac{t}{2}} \Phi(\widetilde{y}, t) e^{\widetilde{y}} \Phi(\widetilde{y}, t)^{-1}, \quad \underline{B}_\infty = \Phi(-\widetilde{y} - t, t) e^a \Phi(\widetilde{y}, t)^{-1}. \tag{2.10}$$

with variables $t = -[a, b]$ and $\tilde{y} = -\frac{\operatorname{ad}(a)}{e^{\operatorname{ad}(a)} - 1}(b) = -\sum_{n \geq 0} \frac{B_n}{n!} \operatorname{ad}^n(a)(b)$.

Proof: This is proved in [4], Section 5.2, using in addition that $\widetilde{\varepsilon}_{2k}([a, b]) = 0$ (2.2).[2] □

The series $\underline{A}_\infty$ and $\underline{B}_\infty$ are the regularized limits of the series $\underline{A}(\tau)$ and $\underline{B}(\tau)$, as $\tau \to i\infty$. In particular, they are also exponentials of formal Lie series, and we define $\underline{\mathfrak{A}}_\infty := \log(\underline{A}_\infty)$, $\underline{\mathfrak{B}}_\infty := \log(\underline{B}_\infty)$.

3 Action of Eisenstein derivations on the meta-abelian quotient

We compute the action of the derivations $\widetilde{\varepsilon}_{2k}$ on the meta-abelian quotient

$$\widehat{\mathcal{L}}^{\mathrm{met-ab}} := \widehat{\mathcal{L}} / \widehat{\mathcal{L}}^{(2)} \tag{3.1}$$

[2] In [4], the series $\underline{A}(\tau)$ and $\underline{B}(\tau)$ are written down in variables x, y, which is why the formulas there look slightly different from the ones given here.

of $\widehat{\mathcal{L}}$. Here, $\widehat{\mathcal{L}}^{(2)}$ is the second term in the derived series of $\widehat{\mathcal{L}}$, i.e. $\widehat{\mathcal{L}}^{(2)} = [\widehat{\mathcal{L}}^{(1)}, \widehat{\mathcal{L}}^{(1)}]$ is the double commutator of $\widehat{\mathcal{L}}$. Note that the quotient $\widehat{\mathcal{L}}^{\mathrm{ab}} := \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(1)} \cong \mathbb{C}a \oplus \mathbb{C}b$ is isomorphic to the abelianization of $\widehat{\mathcal{L}}$, and we have an isomorphism of $\mathbb{C}$-vector spaces

$$\begin{aligned} \widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)} &\to \mathbb{C}[[A,B]] \\ \mathrm{ad}^k(a)\,\mathrm{ad}^l(b)([a,b]) &\mapsto A^k B^l \end{aligned} \tag{3.2}$$

(cf. [3], §2). From this and from the short exact sequence $0 \to \widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)} \to \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)} \to \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(1)} \to 0$, it follows that

$$\widehat{\mathcal{L}}^{\mathrm{met-ab}} \cong \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(1)} \oplus \widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)} \cong (\mathbb{C}a \oplus \mathbb{C}b) \oplus \mathbb{C}[[A,B]] \tag{3.3}$$

as $\mathbb{C}$-vector spaces. As already mentioned in the introduction, for an element $f \in \widehat{\mathcal{L}}$, we will write $f^{\mathrm{met-ab}} = f^{(0)} + f^{(1)}$ for its image in $\widehat{\mathcal{L}}^{\mathrm{met-ab}}$, where $f^{(0)} \in \mathbb{C}a \oplus \mathbb{C}b$ and $f^{(1)} \in \mathbb{C}[[A,B]]$. In order to compute the action of the Eisenstein derivations $\widetilde{\varepsilon}_{2k}$ on $\widehat{\mathcal{L}}^{\mathrm{met-ab}}$, it is enough to compute their action on $\mathbb{C}[[A,B]]$, since we already know their values on a and b by their definition (2.1).

Proposition 3.1. (i) *Identifying $\widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)}$ with $\mathbb{C}[[A,B]]$ as in (3.2), $\widetilde{\varepsilon}_0$ acts on $\widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)}$ as the derivation $-B\frac{\partial}{\partial A}$, while for $k > 0$, $\widetilde{\varepsilon}_{2k}$ acts trivially on $\widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)}$.*

(ii) *Let $\underline{2k} = (2k_1, \ldots, 2k_n)$ be a multi-index, where $k_i \geq 0$. Then $\widetilde{\varepsilon}_{\underline{2k}} = \widetilde{\varepsilon}_{2k_1} \circ \ldots \circ \widetilde{\varepsilon}_{2k_n}$ acts non-trivially on $\widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)}$, if and only if either $\underline{2k} = (0, \ldots, 0, 2k_n)$ or $\underline{2k} = (0, \ldots, 0, 2k_{n-1}, 0)$.*

Proof: By the Jacobi identity, the linear operators $\mathrm{ad}(a), \mathrm{ad}(b) \in \mathrm{Aut}_{\mathbb{Q}}(\widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)})$ commute with each other. Consequently, we have

$$\begin{aligned} \widetilde{\varepsilon}_0(\mathrm{ad}^k(a)\,\mathrm{ad}^l(b)([a,b])) &\equiv \sum_{i=0}^{k-1} -\mathrm{ad}^i(a)\,\mathrm{ad}(b)\,\mathrm{ad}^{k-1-i}(a)\,\mathrm{ad}^l(b)([a,b]) \\ &\equiv -k\,\mathrm{ad}^{k-1}(a)\,\mathrm{ad}^{l+1}(b)([a,b]) \mod \widehat{\mathcal{L}}^{(2)}, \end{aligned} \tag{3.4}$$

and the first part of (i) follows. The triviality of the action of $\widetilde{\varepsilon}_{2k}$ on $\widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)}$, for $k > 0$, follows immediately from the definition of the $\widetilde{\varepsilon}_{2k}$ (2.1) and the fact that they annihilate the commutator $[a,b]$ (2.2). Finally, ii) follows directly from i). □

We now return to the elliptic KZB associator.

Proposition 3.2. *We have*

$$\underline{\mathfrak{A}}(\tau)^{(0)} = -b, \quad \underline{\mathfrak{B}}(\tau)^{(0)} = a - \tau b. \tag{3.5}$$

Proof: Consider the series $\underline{A}_\infty = e^{\frac{t}{2}}\Phi(\tilde{y},t)e^{\tilde{y}}\Phi(\tilde{y},t)^{-1}$ defined in Proposition 2.3. Since $\varphi(x,y) = \log(\Phi(x,y)) \in \widehat{\mathcal{L}}^{(1)}$, applying the Baker-Campbell-Hausdorff formula ([16], Corollary 3.24), and using the fact that $\tilde{y} \equiv -b \mod \widehat{\mathcal{L}}^{(1)}$, we get that

$$\underline{\mathfrak{A}}_\infty = \log(\underline{A}_\infty) \equiv -b \mod \widehat{\mathcal{L}}^{(1)}. \tag{3.6}$$

From Proposition 2.3, we know that $\underline{A}(\tau) = g(\tau)(\underline{A}_\infty) = g(\tau)(\exp(\underline{\mathfrak{A}}(\tau)))$, where $g(\tau) = \sum \mathcal{E}(\underline{2k};\tau)\widetilde{\varepsilon}_{\underline{2k}}$. Since $g(\tau)$ is a continuous automorphism of $\mathbb{C}\langle\langle a,b\rangle\rangle$, it commutes with exponentiation, and we have $\underline{\mathfrak{A}}(\tau) = g(\tau)(\underline{\mathfrak{A}}_\infty)$. Moreover, since $\widetilde{\varepsilon}_0$ annihilates b and all other $\widetilde{\varepsilon}_{2k}$ act trivially on $\widehat{\mathcal{L}}^{\mathrm{ab}}$, we see that

$$g(\tau)(\underline{\mathfrak{A}}_\infty) \equiv -b \mod \widehat{\mathcal{L}}^{(1)}, \tag{3.7}$$

hence $\underline{\mathfrak{A}}(\tau)^{(0)} = -b$. The proof for $\underline{\mathfrak{B}}(\tau)^{(0)}$ is analogous. From the explicit formula $\underline{B}_\infty = \Phi(-\tilde{y}-t,t)e^a\Phi(\tilde{y},t)^{-1}$ and [16], Corollary 3.24, it follows that

$$\underline{\mathfrak{B}}_\infty = \log(\underline{B}_\infty) \equiv a \mod \widehat{\mathcal{L}}^{(1)}. \tag{3.8}$$

The only $\widetilde{\varepsilon}_{\underline{2k}}$, which acts non-trivially on a modulo $\widehat{\mathcal{L}}^{(1)}$, is $\widetilde{\varepsilon}_0$, with $\widetilde{\varepsilon}_0(a) = -b$. Thus, using that $\mathcal{E}(0;\tau) = \tau$, we get

$$\underline{\mathfrak{B}}(\tau) = g(\tau)(\underline{\mathfrak{B}}_\infty) \equiv (\mathrm{id} + (\mathcal{E}(0;\tau)\widetilde{\varepsilon}_0))(a) \equiv a - \tau b \mod \widehat{\mathcal{L}}^{(1)}. \tag{3.9}$$

□

4 Proofs of the main results

In this section, we prove Theorem 1.2, i.e. we compute $\underline{\mathfrak{A}}(\tau)^{(1)}$ and $\underline{\mathfrak{B}}(\tau)^{(1)}$. The plan of the proof is essentially identical to the proof of Proposition 3.2, only technically more involved. To be precise, we first compute $\underline{\mathfrak{A}}_\infty^{(1)}$ and $\underline{\mathfrak{B}}_\infty^{(1)}$ using the Baker-Campbell-Hausdorff formula. This is the content of Section 4.1. In Section 4.3, we then apply the element $g(\tau)$ to $\underline{\mathfrak{A}}_\infty^{(1)}$ and $\underline{\mathfrak{B}}_\infty^{(1)}$, which yields Theorem 1.2.

4.1. The constant term. We will need the following proposition about the Drinfeld associator, which is well-known. It can be deduced for example from (1.4).

Proposition 4.1. *Let* $\varphi(x,y) := \log(\Phi(x,y))$. *Then*

$$\varphi(\tilde{y},t) \equiv -\sum_{n\geq 2} \frac{\zeta(n)}{(-2\pi i)^n} \operatorname{ad}^{n-1}(b)([a,b]) \mod \widehat{\mathcal{L}}^{(2)}, \tag{4.1}$$

where $\tilde{y} = -\frac{\operatorname{ad}(a)}{e^{\operatorname{ad}(a)}-1}(b)$ *and* $t = -[a,b]$.

Theorem 4.2. *We have*

$$\underline{\mathfrak{A}}_\infty^{(1)} = -\left(\sum_{k\geq 2}\lambda_k A^{k-1} + \frac{1}{4}B - \sum_{k\geq 3,\mathrm{odd}} \frac{\zeta(k)}{(2\pi i)^k} B^k\right) \tag{4.2}$$

and

$$\underline{\mathfrak{B}}_\infty^{(1)} = -\left(\sum_{k\geq 2}\lambda_k B^{k-1} + \sum_{k\geq 3,\,\mathrm{odd}} \frac{\zeta(k)}{(2\pi i)^k} AB^{k-1} + \sum_{m,n\geq 2}\lambda_m\lambda_n A^m B^{n-1}\right), \tag{4.3}$$

where $\lambda_k := \frac{B_k}{k!}$, *with* B_k *denoting the* k*-th Bernoulli number.*

Proof: Recall from Proposition 2.3 that $\underline{\mathfrak{A}}_\infty = \log(e^{\frac{t}{2}}\Phi(\tilde{y},t)e^{\tilde{y}}\Phi(\tilde{y},t)^{-1})$. Since $\varphi(\tilde{y},t) \in \widehat{\mathcal{L}}^{(1)}$ by Proposition 4.1, we obtain from [16], Corollary 3.24,

$$S := \log(e^{\frac{t}{2}}\Phi(\tilde{y},t)) \equiv \varphi(\tilde{y},t) + \sum_{k\geq 0}\frac{B_k}{k!}\operatorname{ad}^k(\varphi(\tilde{y},t))\left(\frac{t}{2}\right) \equiv \varphi(\tilde{y},t) + \frac{t}{2} \mod \widehat{\mathcal{L}}^{(2)}. \tag{4.4}$$

Similarly, since $\tilde{y} \equiv -b \mod \widehat{\mathcal{L}}^{(1)}$,

$$\begin{aligned} T := \log(e^{\tilde{y}}\Phi(\tilde{y},t)^{-1}) &\equiv -\log(\Phi(\tilde{y},t)e^{-\tilde{y}}) \equiv \tilde{y} - \sum_{k\geq 0}\frac{B_k}{k!}\operatorname{ad}^k(-\tilde{y})(\varphi(\tilde{y},t)) \\ &\equiv \tilde{y} - \varphi(\tilde{y},t) - \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(b)(\varphi(\tilde{y},t)) \mod \widehat{\mathcal{L}}^{(2)}. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) and again applying [16], Corollary 3.24, we get

$$\begin{aligned} \underline{\mathfrak{A}}_\infty &\equiv T + \sum_{n\geq 0}\frac{B_n}{n!}\operatorname{ad}^n(T)(S) \\ &\equiv \tilde{y} - \varphi(\tilde{y},t) - \sum_{n\geq 1}\frac{B_n}{n!}\operatorname{ad}^n(b)(\varphi(\tilde{y},t)) + \varphi(\tilde{y},t) + \frac{t}{2} + \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(\tilde{y})(\varphi(\tilde{y},t)+\pi i t) \\ &\equiv \tilde{y} + \frac{t}{2} - \sum_{k\geq 1}\frac{B_k}{k!}(\operatorname{ad}^k(b)(\varphi(\tilde{y},t) - (-1)^n\varphi(\tilde{y},t))) + \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(-b)\left(\frac{t}{2}\right) \\ &\equiv \tilde{y} + \frac{t}{2} + \operatorname{ad}(b)(\varphi(\tilde{y},t)) + \sum_{k\geq 1}\frac{(-1)^k}{2}\frac{B_k}{k!}\operatorname{ad}^k(b)(t), \\ &\equiv -b - \sum_{k\geq 2}\frac{B_k}{k!}\operatorname{ad}^{k-1}(a)([a,b]) + \operatorname{ad}(b)(\varphi(\tilde{y},t)) + \sum_{k\geq 1}\frac{(-1)^k}{2}\frac{B_k}{k!}\operatorname{ad}^k(b)(t) \mod \widehat{\mathcal{L}}^{(2)}, \end{aligned} \tag{4.6}$$

where in the second to last line, we have used that $B_1 = -\frac{1}{2}$ and that $B_{2n+1} = 0$ for all $n \geq 1$. Using Proposition 4.1, it follows that

$$\operatorname{ad}(b)(\varphi(\tilde{y},t)) + \sum_{k\geq 1} \frac{(-1)^k}{2}\frac{B_k}{k!}\operatorname{ad}^k(b)(t) = -\sum_{k\geq 2}\frac{\zeta(k)}{(-2\pi i)^k}\operatorname{ad}^k(b)([a,b]) - \sum_{k\geq 1}(-1)^k\frac{B_k}{2k!}\operatorname{ad}^k(b)([a,b]). \tag{4.7}$$

For even $k \geq 2$, we have $-\frac{\zeta(k)}{(-2\pi i)^k} = \frac{B_k}{2k!}$, using Euler's formula for $\zeta(2k)$. Thus (4.7) equals

$$-\frac{1}{4}\operatorname{ad}(b)([a,b]) + \sum_{k\geq 3,\mathrm{odd}}\frac{\zeta(k)}{(2\pi i)^k}\operatorname{ad}^k(b)([a,b]), \tag{4.8}$$

and (4.2), follows from combining (4.6) and (4.8), using the isomorphism (3.2) and noting that the $-b$ term does not contribute to $\underline{\mathfrak{A}}_\infty^{(1)}$. The calculation for $\underline{\mathfrak{B}}_\infty^{(1)}$ is very similar. First, by definition $\underline{\mathfrak{B}}_\infty = \log(\Phi(-\tilde{y}-t,t)e^a\Phi(\tilde{y},t)^{-1})$. Furthermore,

$$\begin{aligned} T := \log(e^a\Phi(\tilde{y},t)^{-1}) \equiv -\log(\Phi(\tilde{y},t)e^{-a}) &\equiv a - \sum_{k\geq 0}\frac{B_k}{k!}\operatorname{ad}^k(-a)(\varphi(\tilde{y},t)) \\ &\equiv a - \varphi(\tilde{y},t) - \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(-a)(\varphi(\tilde{y},t)) \mod \widehat{\mathcal{L}}^{(2)}. \end{aligned} \tag{4.9}$$

We obtain

$$\begin{aligned} \underline{\mathfrak{B}}_\infty \equiv \log(\Phi(-\tilde{y}-t,t)e^T) &\equiv T + \sum_{k\geq 0}\frac{B_k}{k!}\operatorname{ad}^k(T)(\varphi(-\tilde{y}-t,t)) \\ &\equiv T + \varphi(-\tilde{y}-t,t) + \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(a)(\varphi(-\tilde{y}-t,t)) \mod \widehat{\mathcal{L}}^{(2)}, \end{aligned} \tag{4.10}$$

where the last equality follows from the fact that $T \equiv a \mod \widehat{\mathcal{L}}^{(1)}$. Calculating further, we get

$$\begin{aligned} &T + \varphi(-\tilde{y}-t,t) + \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(a)(\varphi(-\tilde{y}-t,t)) \\ &\equiv a - \varphi(\tilde{y},t) + \varphi(-\tilde{y}-t,t) - \sum_{k\geq 1}\frac{B_k}{k!}(-1)^k\operatorname{ad}^k(a)(\varphi(\tilde{y},t)) + \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(a)(\varphi(-\tilde{y}-t,t)) \\ &\equiv a - \varphi(\tilde{y},t) + \varphi(-\tilde{y}-t,t) - \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(a)\left[(-1)^k\varphi(\tilde{y},t)) - \varphi(-\tilde{y}-t,t))\right] \\ &\equiv a + \sum_{k\geq 2}\frac{-\zeta(k)}{(-2\pi i)^k}\operatorname{ad}^{k-1}(b)(-[a,b]) + \sum_{k\geq 2}\frac{-\zeta(k)}{(2\pi i)^k}\operatorname{ad}^{k-1}(b)(-[a,b]) \\ &\quad - \sum_{k\geq 1}\frac{B_k}{k!}\operatorname{ad}^k(a)\left[(-1)^k\sum_{n\geq 2}\frac{\zeta(n)}{(-2\pi i)^n}\operatorname{ad}^{n-1}(b)(-[a,b]) - \sum_{k\geq 2}\frac{-\zeta(k)}{(2\pi i)^k}\operatorname{ad}^{k-1}(b)(-[a,b])\right] \\ &\equiv a + \sum_{k\geq 2}\left[1-(-1)^{k-1}\right]\frac{\zeta(k)}{(-2\pi i)^k}\operatorname{ad}^{k-1}(b)([a,b]) + \frac{1}{2}\left[-\sum_{k\geq 2}(-1)^{k-1}+1)\frac{\zeta(k)}{(2\pi i)^k}\right]\operatorname{ad}(a)\operatorname{ad}^{k-1}(b)([a,b]) \\ &\quad - \sum_{m\geq 2,\,\mathrm{even}}\frac{B_m}{m!}\left[\sum_{n\geq 2}\left[(-1)^{n-1}-1\right]\frac{\zeta(n)}{(2\pi i)^n}\right]\operatorname{ad}^m(a)\operatorname{ad}^{n-1}(b)([a,b]) \mod \widehat{\mathcal{L}}^{(2)}. \end{aligned} \tag{4.11}$$

Again using that $-\frac{\zeta(k)}{(-2\pi i)^k} = \frac{B_k}{2k!}$, if $k \geq 2$ is even, we obtain from (4.11)

$$\begin{aligned} &-\sum_{k\geq 2}\frac{B_k}{k!}\operatorname{ad}^{k-1}(b)([a,b]) - \sum_{k\geq 3,\,\mathrm{odd}}\frac{\zeta(k)}{(2\pi i)^k}\operatorname{ad}(a)\operatorname{ad}^{k-1}(b)([a,b]) \\ &\qquad - \sum_{m\geq 2}\frac{B_m}{m!}\sum_{n\geq 2}\frac{B_n}{n!}\operatorname{ad}^m(a)\operatorname{ad}^{n-1}(b)([a,b]) \end{aligned}$$

$$\equiv a - \Bigg[\sum_{k\geq 2} \frac{B_k}{k!} \operatorname{ad}^{k-1}(b)([a,b]) + \sum_{k\geq 3,\, \mathrm{odd}} \frac{\zeta(k)}{(2\pi i)^k} \operatorname{ad}(a) \operatorname{ad}^{k-1}(b)([a,b]) \\ + \sum_{m,n\geq 2} \frac{B_m B_n}{m!n!} \operatorname{ad}^m(a) \operatorname{ad}^{n-1}(b)([a,b]) \Bigg] \mod \widehat{\mathcal{L}}^{(2)}. \tag{4.12}$$

The first term a belongs to the abelianization, and does not contribute to $\underline{\mathfrak{B}}_\infty^{(1)}$. By applying the isomorphism $\widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)} \cong \mathbb{C}[[A,B]]$ to the remaining terms in (4.12), we obtain the desired result (4.3). □

4.2. The relation to period polynomials. We now show how Theorem 4.2 implies that $\mathfrak{A}_\infty^{(1)}$ and $\mathfrak{B}_\infty^{(1)}$ can be interpreted as generating series for the extended period polynomials of the Eisenstein series E_{2k} [20]. Recall that for $k \geq 2$, the extended period polynomial $r_{E_{2k}}(A,B)$ of E_{2k} equals

$$\omega^+_{E_{2k}}(A^{2k-2} - B^{2k-2}) + \omega^-_{E_{2k}} \sum_{-1\leq n\leq 2k-1} \lambda_{n+1}\lambda_{2k-1-n} A^n B^{2k-2-n} \in \bigoplus_{-1\leq n\leq 2k-1} \mathbb{C}\cdot A^n B^{2k-2-n}, \tag{4.13}$$

where $\lambda_k = \frac{B_k}{k!}$ and the numbers $\omega^\pm_{E_{2k}} \in \mathbb{C}$ are given by $\omega^+_{E_{2k}} = \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}\omega^-_{E_{2k}}$, $\omega^-_{E_{2k}} = -\frac{(2k-2)!}{2}$. We caution the reader that, despite its name, $r_{E_{2k}}$ is not a polynomial. Define

$$\widetilde{\mathfrak{A}}(A,B) = \frac{1}{B}\underline{\mathfrak{A}}_\infty^{(1)}(A,B), \quad \widetilde{\mathfrak{B}}(A,B) = \frac{1}{A}\underline{\mathfrak{B}}_\infty^{(1)}(A,B). \tag{4.14}$$

Both are elements of $(\mathbb{C}[[A,B]])[A^{-1},B^{-1}]$. Now, for $f \in \mathbb{C}[[A,B]][A^{-1},B^{-1}]$ we denote by f_k its homogeneous component of degree k. We also denote by $f_k = f_k^+ + f_k^-$ the decomposition of f_k into its even and odd parts in A. Write $\widetilde{\mathfrak{A}} = \sum_{k\geq 1} \widetilde{\mathfrak{A}}_{2k-2}$, $\widetilde{\mathfrak{B}} = \sum_{k\geq 1} \widetilde{\mathfrak{B}}_{2k-2}$. Comparing the explicit formula for the extended period polynomials (4.13) with Theorem 4.2, it follows that for $k \geq 2$,

$$r_{E_{2k}}(A,B) = \frac{(2k-2)!}{2}\left(\widetilde{\mathfrak{A}}(A,B)^+_{2k-2} + \widetilde{\mathfrak{B}}(B,A)^+_{2k-2} + \widetilde{\mathfrak{A}}(A,B)^-_{2k-2} + \widetilde{\mathfrak{B}}(A,B)^-_{2k-2}\right). \tag{4.15}$$

This proves Theorem 1.3. Moreover, it also suggests a "definition" (which may or may not be new) of the extended period polynomial for the quasi-modular form $E_2(\tau) := -\frac{1}{24} + \sum_{n\geq 1}\sigma_1(n)q^n$, namely:

$$r_{E_2}(A,B) := \frac{1}{2}\left(\widetilde{\mathfrak{A}}(A,B)^+_0 + \widetilde{\mathfrak{B}}(B,A)^+_0 + \widetilde{\mathfrak{A}}(A,B)^-_0 + \widetilde{\mathfrak{B}}(A,B)^-_0\right) \\ = \frac{1}{2}\left(\frac{1}{4} + \frac{\lambda_2}{2}\frac{A}{B} + \frac{\lambda_2}{2}\frac{B}{A}\right) = \frac{1}{24}\left(3 + \frac{A}{B} + \frac{B}{A}\right). \tag{4.16}$$

4.3. The non-constant term. We now complete the proof of Theorem 1.2.

Proof: (of Theorem 1.2) We first prove i). By Proposition 2.3, $\underline{A}_\infty = g(\tau)(\underline{A}_\infty) = g(\tau)(\exp(\underline{\mathfrak{A}}_\infty))$, where $g(\tau) = \sum \mathcal{E}(\underline{2k};\tau)\widetilde{\varepsilon}_{\underline{2k}}$. Since $g(\tau)$ is a continuous automorphism of $\mathbb{C}\langle\langle a,b\rangle\rangle$, we have $\underline{\mathfrak{A}}(\tau) = g(\tau)(\underline{\mathfrak{A}}_\infty)$ and it follows that

$$\underline{\mathfrak{A}}(\tau)^{\mathrm{met-ab}} \equiv g(\tau)(\underline{\mathfrak{A}}_\infty) \mod \widehat{\mathcal{L}}^{(2)}, \tag{4.17}$$

Now by Proposition 3.1 we see that the only multi-indices $\underline{2k} = (2k_1,\dots,2k_n)$ such that $\widetilde{\varepsilon}_{\underline{2k}}$ can possibly act non-trivially on $\underline{\mathfrak{A}}_\infty^{\mathrm{met-ab}}$ are given by $(0,\dots,0,2k)$ and $(0,\dots,0,2k,0)$. But since $\underline{\mathfrak{A}}_\infty^{\mathrm{met-ab}}$ contains no linear term in a by Proposition 3.2, we see that only multi-indices of the shape $(0,\dots,0,2k)$ with $k \geq 0$ can act non-trivially on $\underline{\mathfrak{A}}_\infty^{\mathrm{met-ab}}$, and using Theorem 4.2, we get

$$g(\tau)(\underline{\mathfrak{A}}_\infty^{\mathrm{met-ab}}) \equiv \underline{\mathfrak{A}}_\infty^{\mathrm{met-ab}} + \sum_{n\geq 1} \mathcal{E}(\{0\}_n;\tau)\widetilde{\varepsilon}_0^n\left(\sum_{k\geq 2}\frac{-B_k}{k!}\operatorname{ad}^{k-1}(a)([a,b])\right) \\ + \sum_{n\geq 1,\, k\geq 1} \mathcal{E}(\{0\}_{n-1},2k;\tau)(\widetilde{\varepsilon}_0^{n-1}\circ\widetilde{\varepsilon}_{2k})(-b)$$

$$
\begin{aligned}
&\equiv \underline{\mathfrak{A}}_\infty^{\mathrm{met-ab}} + \sum_{n\geq 1} \mathcal{E}(\{0\}_n;\tau) \sum_{k\geq 1} \frac{-B_{2k}}{(2k)!} \widetilde{\varepsilon}_0^n(\mathrm{ad}^{2k-1}(a)([a,b])) \\
&\quad - \sum_{n\geq 1,\, k\geq 1} \frac{2}{(2k-2)!} \mathcal{E}(\{0\}_{n-1}, 2k;\tau) \widetilde{\varepsilon}_0^{n-1}(\mathrm{ad}^{2k-2}(a)\,\mathrm{ad}(b)([a,b])) \qquad (4.18)\\
&\equiv \underline{\mathfrak{A}}_\infty^{\mathrm{met-ab}} + \sum_{n\geq 1, k\geq 1} \frac{2}{(2k-2)!} \mathcal{E}(\{0\}_n;\tau) \frac{B_{2k}}{4k} \widetilde{\varepsilon}_0^{n-1}(\mathrm{ad}^{2k-2}(a)\,\mathrm{ad}(b)([a,b])) \\
&\quad - \sum_{n\geq 1,\, k\geq 1} \frac{2}{(2k-2)!} \mathcal{E}(\{0\}_{n-1}, 2k;\tau) \widetilde{\varepsilon}_0^{n-1}(\mathrm{ad}^{2k-2}(a)\,\mathrm{ad}(b)([a,b])) \mod \widehat{\mathcal{L}}^{(2)}.
\end{aligned}
$$

Under the isomorphism $\widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)}$ of (3.2), $\mathrm{ad}^r(a)\,\mathrm{ad}^s(b)([a,b])$ is mapped to A^rB^s and by Proposition 3.1, the derivation $\widetilde{\varepsilon}_0$ corresponds to the derivation $-B\frac{\partial}{\partial A}$ of $\mathbb{C}[[A,B]]$. Therefore, writing $\underline{\mathfrak{A}}(\tau)^{\mathrm{met-ab}} = \underline{\mathfrak{A}}(\tau)^{(0)} + \underline{\mathfrak{A}}(\tau)^{(1)}$ and setting $m = 2k-1-n$, we have

$$
\underline{\mathfrak{A}}(\tau)^{(1)} = \underline{\mathfrak{A}}_\infty^{(1)} + \sum_{m\geq 0,\, n\geq 1} \frac{2}{(m+n-1)!} \alpha_{m,n}(\tau) \left(-B\frac{\partial}{\partial A}\right)^{n-1} A^{m+n-1}B, \qquad (4.19)
$$

where $\alpha_{m,n}(\tau) = -\mathcal{E}(\{0\}_{n-1}, m+n+1;\tau) + \frac{B_{m+n+1}}{2(m+n+1)}\mathcal{E}(\{0\}_n;\tau)$, This proves i).

The proof of ii) is similar. First, recall from Theorem 4.2 that

$$
\underline{\mathfrak{B}}_\infty^{(0)} = a, \quad \underline{\mathfrak{B}}_\infty^{(1)} = -\sum_{m,n\geq 0} c_{m,n}\,\mathrm{ad}^m(a)\,\mathrm{ad}^n(b)([a,b]) \mod \widehat{\mathcal{L}}^{(2)}, \qquad (4.20)
$$

where $c_{m,n}$ is the coefficient of A^mB^n in $\underline{\mathfrak{B}}_\infty^{(1)}$. Again by Proposition 3.1, the only $\widetilde{\varepsilon}_{\underline{2k}}$, which act non-trivially on $\underline{\mathfrak{B}}_\infty^{\mathrm{met-ab}}$ are the ones whose associated multi-index is equal to $(0,\ldots,0,2k)$ or to $(0,\ldots,0,2k,0)$. Using the explicit formulas for $\underline{\mathfrak{B}}_\infty^{(0)}$ (3.5) and for $\underline{\mathfrak{B}}_\infty^{(1)}$ (4.3) on the other hand, we get that

$$
\begin{aligned}
g(\tau)(\underline{\mathfrak{B}}_\infty^{\mathrm{met-ab}}) \equiv \underline{\mathfrak{B}}_\infty^{\mathrm{met-ab}} + \mathcal{E}(0;\tau)\widetilde{\varepsilon}_0(a) - \sum_{r\geq 1} \mathcal{E}(\{0\}_r;\tau)\widetilde{\varepsilon}_0^r \left(\sum_{m,n\geq 0} c_{m,n}\,\mathrm{ad}^m(a)\,\mathrm{ad}^n(b)([a,b])\right) \\
+ \sum_{k\geq 1} \mathcal{E}(\{0\}_{r-1}, 2k;\tau)(\widetilde{\varepsilon}_0^{r-1}\circ\widetilde{\varepsilon}_{2k})(a) \\
+ \sum_{k\geq 1} \mathcal{E}(\{0\}_{r-2}, 2k, 0;\tau)(\widetilde{\varepsilon}_0^{r-2}\circ\widetilde{\varepsilon}_{2k}\circ\widetilde{\varepsilon}_0)(a) \mod \widehat{\mathcal{L}}^{(2)}. \quad (4.21)
\end{aligned}
$$

Using (3.2) together with Proposition 3.1 and writing $\underline{\mathfrak{B}}(\tau)^{\mathrm{met-ab}} = \underline{\mathfrak{B}}(\tau)^{(0)} + \underline{\mathfrak{B}}(\tau)^{(1)}$, we finally obtain

$$
\begin{aligned}
\underline{\mathfrak{B}}(\tau)^{(1)} = \underline{\mathfrak{B}}_\infty^{(1)} - \sum_{r\geq 1} \mathcal{E}(\{0\}_r;\tau) \left(\sum_{m,n\geq 0} c_{m,n} \left(-B\frac{\partial}{\partial A}\right)^r A^mB^n\right) \\
+ \sum_{k\geq 1} \frac{2}{(2k-2)!} \mathcal{E}(\{0\}_{r-1}, 2k;\tau) \left(-B\frac{\partial}{\partial A}\right)^{r-1} A^{2k-1} \\
- \sum_{k\geq 1} \frac{2}{(2k-2)!} \mathcal{E}(\{0\}_{r-2}, 2k, 0;\tau) \left(-B\frac{\partial}{\partial A}\right)^{r-2} A^{2k-2}B, \qquad (4.22)
\end{aligned}
$$

and (ii) follows. □

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Zusammenfassung

Diese Arbeit beschäftigt sich mit einem elliptischen Analogon der multiplen Zeta Werte, den elliptischen multiplen Zeta Werten. Multiple Zeta Werte sind Verallgemeinerungen der speziellen Werte der Riemannschen Zetafunktion, und lassen sich als homotopie-invariante iterierte Integrale auf $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ schreiben. In ähnlicher Weise sind elliptische multiple Zeta Werte gegeben durch homotopie-invariante iterierte Integrale auf einer einfach punktierten elliptischen Kurve.

Ziel dieser Arbeit war die Algebra der elliptischen multiplen Zeta Werte zu studieren. Da es auf einer elliptischen Kurve zwei natürliche Homologie-Zyklen gibt, existieren zwei Algebren $\mathcal{EZ}^{\mathrm{A}}$ und $\mathcal{EZ}^{\mathrm{B}}$ von elliptischen multiplen Zeta Werten, welche durch eine modulare Transformation miteinander verbunden sind.

Per Definition sind die Erzeuger der $\mathbb{Q}$-Algebra $\mathcal{EZ}^{\mathrm{A}}$ gegeben durch die A-elliptischen multiplen Zeta Werte. Eine der grundlegenden Arbeitshypothesen ist, dass die einzigen Relationen in der $\mathbb{Q}$-Algebra $\mathcal{EZ}^{\mathrm{A}}$ gegeben sind durch shuffle und Fay Relationen. Darauf aufbauend wird die Längenfiltrierung $\mathcal{L}(\mathcal{EZ}^{\mathrm{A}})$ auf $\mathcal{EZ}^{\mathrm{A}}$ studiert, und die Zahlen $D^{ell}_{k,n} := \mathcal{L}_n(\mathcal{EZ}^{\mathrm{A}}_k)/\mathcal{L}_{n-1}(\mathcal{EZ}^{\mathrm{A}}_k)$ studiert. Für $n \leq 2$ wird eine explizite Formel bewiesen, während im Falle $n = 3$ ein Teilresultat erzielt werden.

Es wird gezeigt, dass die Algebra der elliptischen multiplen Zeta Werte in eine nicht-kommutative Polynomalgebra in den Variablen $\mathbf{e}_{2k}$, mit $k \geq 0$, eingebettet werden kann. Es werden Resultate erzielt, welche das Bild dieser Einbettung genauer spezifizieren.

Es wird die Erzeugendenreihe der elliptischen multiplen Zeta Werte, der elliptische KZB Assoziator studiert. Genauer berechnen wir das Bild des elliptischen KZB Assoziators im meta-abelschen Quotienten der freien Lie Algebra auf zwei Erzeugern, und erhalten eine Verbindung zu Periodenpolynomen von Eisensteinreihen

Es wird gezeigt, dass elliptische multiple Zeta Werte in Berechnungen der String Theorie erscheinen. Genauer wird gezeigt, dass die offene Superstring Amplitude für eine Schleife durch elliptische multiple Zeta Werte ausgedrückt werden kann.

Abstract

The topic of this thesis is the study of an elliptic analogue of multiple zeta values, the elliptic multiple zeta values. Multiple zeta values are generalizations of special values of the Riemann zeta function, which can be written as homotopy invariant iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. In a similar way, elliptic multiple zeta values are given by homotopy invariant iterated integrals on a once-punctured elliptic curve.
The goal of this work was to study the algebra of elliptic multiple zeta values. Since there are two natural homology cycles on an elliptic curve, there exist two algebra $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$ of elliptic multiple zeta values, which are related to each other by a simple modular transformation.
By definition, the $\mathbb{Q}$-algebra $\mathcal{EZ}^{\mathrm{A}}$ is generated by the A-elliptic multiple zeta values. One of the main working hypotheses is that the only relations between these generators are given by the shuffle and Fay relations. Building on this hypothesis, we define the length filtration $\mathcal{L}(\mathcal{EZ}^{\mathrm{A}})$, and study the numbers $D^{ell}_{k,n} := \mathcal{L}_n(\mathcal{EZ}^{\mathrm{A}}_k)/\mathcal{L}_{n-1}(\mathcal{EZ}^{\mathrm{A}}_k)$. For $n \leq 2$, we prove an explicit formula, and for $n = 3$, we obtain partial results. These results confirm our working hypothesis in lengths $n \leq 3$.
We show that the algebra of elliptic multiple zeta values embeds into an algebra of non-commutative polynomials in variables $\mathbf{e}_{2k}$, for $k \geq 0$, and we prove results which specify the image of this embedding.
We study the generating series of elliptic multiple zeta values, known as the elliptic KZB associator. More precisely, we compute the image of the elliptic KZB associator in the meta-abelian quotient of a free Lie algebra on two generators, and show that this image is related to period polynomials of Eisenstein series.
We show that elliptic multiple zeta values occur in string theory computations. More precisely, we show that the open superstring amplitude at one-loop level can be expressed in terms of elliptic multiple zeta values.

Erklärung zum Eigenanteil

Die Inhalte von Appendix C und D meiner Dissertation "Elliptic multiple zeta values" sind veröffentlichte Artikel, die in gemeinsamer Arbeit entstanden.

i) Elliptic multiple zeta values and one-loop open superstring amplitudes ([13])
Zusammenarbeit mit: Johannes Broedel, Carlos Mafra und Oliver Schlotterer
Der Grundstein für diese Arbeit wurde gelegt durch Diskussionen mit Oliver Schlotterer nach meinem Vortrag bei der Konferenz "Numbers and physics" am ICMAT in Madrid im September 2014. In diesen Gespraechen und nachfolgenden Treffen mit Broedel, Mafra und Schlotterer am AEI Potsam im Oktober 2014 wurde klar, dass die von mir studierten elliptischen multiplen Zeta Werte benutzt werden können um gewisse Ein-Schleifen String Amplituden auszudrücken und weiter zu studieren. Ziel der gemeinsamen Arbeit [13] war es diese Erkenntnisse darzustellen.
Ein großer Teil des dritten Abschnitts von [13] wurde von mir geschrieben, und ich habe im ganzen Artikel viele mathematische Anregungen und Vorschläge gemacht. Gerade der Bezug zur Physik und die konkreten Berechnungen zur String Amplitude in [13] wären aber ohne die Expertise von Broedel, Mafra und Schlotterer nicht möglich gewesen.

ii) Elliptic multiple zeta values and a special derivation algebra ([14])
Zusammenarbeit mit: Johannes Broedel und Oliver Schlotterer
Die Idee für diese Arbeit enstand in gemeinsamen Diskussionen mit Broedel, Mafra und Schlotterer bei einem Besuch in Cambridge im Februar 2015. Ziel war es gewisse Strukturen elliptischer multipler Zeta Werte, welche aus dem Kontext des vorherigen Artikels [13] enstanden weiter zu untersuchen.
Ein Kernstück der Arbeit bilden explizite Berechnungen von elliptischen multiplen Zeta Werten, welche dabei helfen die Dimensionen gewisser durch diese aufgespannten Vektorräumen zu bestimmen. Einige dieser Berechnungen wurden im Wesentlichen von mir konzipiert, und dann in gemeinsamer Arbeit implementiert und erweitert. Das Aufschreiben der Arbeit erfolgte dann gemeinsam, und wurde weitestgehend während eines gemeinsamen Forschungsaufenthaltes am MITP in Mainz im Mai-Juni 2015 abgeschlossen.

Eidesstattliche Erklärung

Ich versichere hiermit an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe.

Hamburg, 01.11.2016

Lebenslauf

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