AROUND INTEGRALS OF MODULAR FORMS FOR $\mathrm{SL}_2(\mathbb{Z})$

NILS MATTHES

ABSTRACT. These are expanded lecture notes for a course aimed at graduate students given in February 2019 at ETH Zurich as part of the programme *Modular forms, periods and scattering amplitudes*. They are intended as a concise first introduction to periods of modular forms in a manner accessible to mathematicians and physicists alike.

Warning for the reader: These are informal lecture notes, so beware of typos and mistakes. If you find any, please let me know under:

nils.matthes@maths.ox.ac.uk.

CONTENTS

1. Introduction

1.1. **Motivation.** Many special functions which initially are defined on subsets of the real line $\mathbb{R}$ can be extended to holomorphic functions on the complex plane $\mathbb{C}$. Besides trivial examples such as polynomials, the best known example is arguably the exponential function $z \mapsto \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Other functions such as the Gamma function $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\mathrm{d}t$ or the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ can at least be extended meromorphically, i.e. they extend to holomorphic functions outside certain discrete sets of poles. In fact both functions only have simples poles and there are explicit formulas for the corresponding residues.

The situation is completely different for the real logarithm

$$\log(t) := \int_1^t \frac{\mathrm{d}x}{x}, \quad t > 0,$$

which cannot be extended to a meromorphic function on $\mathbb{C}$. This is a consequence of Cauchy's integral formula

$$\int_{\sigma_0} \frac{\mathrm{d}x}{x} = 2\pi i,$$

where σ_0 is any loop which winds once around 0 in the positive direction. Indeed, if there were a continuous function $F : \mathbb{C}^\times \to \mathbb{C}$ such that $\mathrm{d}F = \frac{\mathrm{d}z}{z}$, then the integral of $\frac{\mathrm{d}z}{z}$ along any closed loop would have to vanish, by the fundamental theorem of calculus.[1] Therefore, the number $2\pi i$ (which is also called a monodromy period in this context) provides an obstruction to existence of a primitive.

On the other hand, there do exist real analytic functions $\mathbb{C}^\times \to \mathbb{C}$ which restrict to the real logarithm on $(0, \infty)$. Among all such extensions, the function $z \mapsto \log|z|$ is singled out as the solution to the differential equation

$$\mathrm{d}F = \mathrm{Re}\left(\frac{\mathrm{d}z}{z}\right) = \frac{1}{2}\left(\frac{\mathrm{d}z}{z} + \frac{\mathrm{d}\bar{z}}{\bar{z}}\right).$$

Intuitively, $\log|z| = \mathrm{Re}(\log(z))$ is well-defined because the period $2\pi i$ of the logarithm is purely imaginary and can therefore be eliminated by taking the real part.

[1] On the other hand, $\log(z)$ can be defined as an analytic function on the half-cut plane $\mathbb{C}\backslash(-\infty, 0]$ or more generally on any open subset $U \subset \mathbb{C} \setminus \{0\}$ which is *simply connected*.

1.2. **Goals and scope.** The goal of these lecture notes is to study an analogous picture for modular forms in the simplest case of forms for $\mathrm{SL}_2(\mathbb{Z})$. More precisely, we will introduce periods of modular forms which will be analogues of the number $2\pi i$ arising as monodromy of the logarithm. These are very rich objects for which there exists a whole theory, *Eichler–Shimura–Manin theory*, and we do not attempt to give a systematic account. Instead, we will content ourselves with a few key results. One of our main points here is that the study of periods of modular forms has concrete implications for modular forms themselves.

Having studied periods of modular forms to some extent, we turn to the construction of modular invariant primitives of modular forms, i.e. the analogues of the function $z \mapsto \log |z|$. Again, the presence of periods implies that these primitives will have to live in a subspace of real analytic rather than holomorphic functions. Moreover, in these notes, we only describe the construction of modular primitives of Eisenstein series in which case the corresponding primitives turn out to be variants of classical real analytic Eisenstein series. On the other hand, the construction of modular primitives of cusp forms is much more involved, both conceptually and technically. It was achieved only recently, [6], and we only briefly sketch some of their main interesting properties towards the very end of these notes.

1.3. **Contents.** Section 1 provides a quick review of modular forms for $\mathrm{SL}_2(\mathbb{Z})$. Many basic statements are given but proofs have been relegated to the literature. In Section 2, we define periods of modular forms and study their relation to L-series. One of the main results on periods is given in Section 4, the Eichler–Shimura theorem which characterizes modular forms algebraically in terms of the associated period polynomials. The Rankin–Selberg method forms the core of Section 4 in which we also give an application of the theory developed so far to the structure of spaces of modular forms. In Section 5 we construct modular invariant primitives of Eisenstein series and relate them to real analytic Eisenstein series. We end by giving some idea how to go about the construction of modular invariant primitives for cusp forms.

Finally, there are two appendices. The first one presents a classical and quite powerful method for computing the Fourier expansions of Eisenstein series, namely the Poisson summation formula. In the second appendix, and with the intention of guiding the reader towards modular forms for more general groups, we rephrase parts of the theory of periods for modular forms in terms of group cohomology.

Acknowledgements Very many thanks to Giovanni Felder for the invitation to give these lectures and to all of its participants for the congenial and stimulating atmosphere in which they were given. It is the author's pleasure to acknowledge the huge intellectual debt these notes owe to the writings of Francis Brown and Don Zagier about modular forms and their periods. The author would also like to thank Deepak Kamlesh for corrections.

2. Quick review of modular forms for $\mathrm{SL}_2(\mathbb{Z})$

We begin by recalling the definition of modular form and of some basic results. We then discuss two extra structures on the space of modular forms, namely the Petersson inner product and the action of a commuting family of endomorphisms, the Hecke operators. For more extensive introductions, see for example [15, Ch. VII] and [22].

2.1. **The notion of modular form.** Before defining modular forms, we need to set up some basic notation. A key role is played by the group $\mathrm{SL}_2(\mathbb{Z})$ of integer 2×2 matrices with unit determinant or rather its action on the extended complex plane $\mathbb{C}\cup\{\infty\}$ via Möbius transformations,

$$\mathrm{SL}_2(\mathbb{Z})\times\mathfrak{H}\to\mathfrak{H},\quad (\gamma,z)\mapsto\gamma.z:=\frac{az+b}{cz+d},\quad \text{where } \gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

Clearly γ gives the identity transformation if and only if $\gamma=\pm I$ where $I\in\mathrm{SL}_2(\mathbb{Z})$ is the identity matrix. Moreover, one can show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ whose associated Möbius transformations are the translation $z\mapsto z+1$ and the inversion $z\mapsto -1/z$ respectively.

Now let $\mathfrak{H}=\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)>0\}$ of complex numbers with positive imaginary part. Since $\mathrm{Im}(\gamma.z)=\mathrm{Im}(z)/|cz+d|^2$ for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, the action of $\mathrm{SL}_2(\mathbb{Z})$ stabilizes $\mathfrak{H}$, and a fundamental domain for this action is given by the set

$$\mathcal{F}=\{z\in\mathbb{C}\,|\,|\mathrm{Re}(z)|<1/2,\ |z|>1\}.$$

By the latter we mean that for every point $z\in\mathfrak{H}$ there exists $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ such that $\gamma.z\in\overline{\mathcal{F}}$, the closure of $\mathcal{F}$ in $\mathfrak{H}$, and if $\gamma_1.z,\gamma_2.z\in\mathfrak{H}$ for $\gamma_1\neq\gamma_2$, then $z\in\partial\mathcal{F}$ the boundary of $\mathcal{F}$.

We can now give the definition of modular forms.

Definition 2.1. A *modular form* of weight $k\in\mathbb{Z}$ for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f:\mathfrak{H}\to\mathbb{C}$ which satisfies

$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k f(z),\quad \text{for all } \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}), \tag{1}$$

and such that $f(x+iy)$ is bounded as $y\to\infty$. We say that f is a *cusp form* if $\lim_{y\to\infty}f(x+iy)=0$.

Before proceeding with examples, we record two consequences of the definition. Firstly, applying (1) with $\gamma=-I$ we see that $f(z)=(-1)^k f(z)$, so that every modular form of odd weight is identically zero. Secondly, by applying (1) with $\gamma=T$ has a Fourier expansion

$$f(z)=\sum_{n\in\mathbb{Z}}^{\infty}a_nq^n,\quad \text{where } q=e^{2\pi iz},$$

and the boundedness condition implies that $a_n=0$ for all $n<0$. Also, note that f is a cusp form if and only if $a_0=0$.

2.2. **Examples of modular forms.**

(i) For an integer $k\geq 3$, consider the Eisenstein series

$$G_k(z)=\frac{(k-1)!}{2(2\pi i)^k}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(mz+n)^k},$$

where $'$ indicates that $m=n=0$ is omitted from the summation. It is a modular form of weight k which vanishes identically if k is odd. For even weight $2k$, its Fourier expansion is given by

$$G_{2k}(z)=-\frac{B_{2k}}{4k}+\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n,$$

where B_n are the Bernoulli numbers and $\sigma_l(n) = \sum_{d|n} d^l$.

(ii) Define the *discriminant function* Δ to be

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad \text{where as before } q = e^{2\pi i z}.$$

This is a cusp form of weight 12 whose Fourier expansion is written as

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n.$$

From the definition of $\Delta(z)$, one sees that the coefficients $\tau(n)$ are integers, and the function $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto \tau(n)$ is known as *Ramanujan's tau-function.*

Note that in both examples we have $a_1 = 1$ which was the reason for including $\frac{(k-1)!}{2(2\pi i)^k}$ in the definition of G_k. The significance of this will become clearer later when we discuss the action of Hecke operators.

2.3. **Growth of Fourier coefficients.** It is not hard to see that the Fourier coefficients $a_n = \sigma_{2k-1}(n)$ of Eisenstein series are bounded by

$$n^{2k-1} \le a_n \le \zeta(2k-1)n^{2k-1}.$$

In particular, there exists a constant C such that $|a_n| \le Cn^{2k-1}$ for all n, and this bound is optimal. There are similar bounds for Fourier coefficients of cusp forms, however these grow much slower.

Proposition 2.2 (Hecke). *Let $f(z) = \sum_{n=1}^{\infty} a_n q^n$ be a cusp form of weight k. Then there exists a real number $C > 0$ such that $|a_n| \le Cn^{k/2}$ for all n.*

Proof. See [15], VII.4.3, Proposition 9 and Theorem 5, or [22], 2.B, Theorem 4.(i). □

The above estimate for Fourier coefficients of cusp forms of weight k is not optimal. In fact, we know that $|a_n| \le \sigma_0(n)n^{(k-1)/2}$ for every n which implies in particular that $|a_n| = O(n^{(k-1)/2+\varepsilon})$ for every $\varepsilon > 0$. However, this better bound lies much deeper; it was proved by Deligne as a consequence of his proof of the analogue of the Riemann hypothesis for varieties over finite fields.

2.4. **Dimension formulas.**

Theorem 2.3. *Let M_k be the $\mathbb{C}$-vector space of modular forms of weight k.*

(i) *We have $M_k = \{0\}$ if $k < 0$ or k is odd, and if $k \ge 0$ is even, then*

$$\dim M_k = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2 \mod 12, \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & k \not\equiv 2 \mod 12. \end{cases}$$

(ii) *A basis for M_k is given by the set $\{G_4(z)^m G_6(z)^n \mid 4m + 6n = k\}$.*

Proof. See [15], VII.3.2, Theorem 4 and Corollary 2. □

The fact that the dimensions of the space of modular forms of a given weight are known explicitly is useful for proving identities between modular forms and leads to the following result, known as *Sturm bound*, [17]. Given modular forms

$f, g \in M_k$ with Fourier expansions $f = \sum_{n=0}^{\infty} a_n q^n$, $g = \sum_{n=0}^{\infty} b_n q^n$, if $a_i = b_i$ for all $0 \leq i \leq k/12$, then $f = g$. For example, the identity

$$\Delta(z) = 8000 \cdot G_4(z)^3 - 147 \cdot G_6(z)^2.$$

can be verified by showing that the zeroth and first Fourier coefficients of both sides are equal.

2.5. **The Petersson inner product.** The space of cusp forms of weight k is endowed with a non-degenerate, positive, Hermitian inner product as follows.

Definition 2.4. Given two modular forms $f, g \in M_k$ such that at least one of f and g is cuspidal, define their *Petersson inner product* $\langle f, g \rangle$ to be

$$\langle f, g \rangle = \iint_{\mathcal{F}} f(z)\overline{g(z)} y^k \mathrm{d}\mu,$$

where $d\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$.

This is well-defined because firstly the integrand is $\mathrm{SL}_2(\mathbb{Z})$-invariant, and secondly the integral converges since the integrand decays exponentially fast as $y \to \infty$, because at least one of f or g is a cusp form.

We will see later that the Petersson inner product can be extended naturally to a non-degenerate and Hermitian (but not always positive definite) inner product on M_k.

2.6. **Hecke operators.** The space M_k carries an action of a certain family of linear operators $\{T(n) : M_k \to M_k\}_{n \geq 1}$, the *Hecke operators.* Their action on the Fourier expansion of a modular form $\sum_{m=0}^{\infty} a_m q^m$ of weight k is given by[2]

$$T(n)f = \sum_{m=0} \alpha_m q^m, \quad \text{where } \alpha_m = \sum_{d|(m,n)} d^{k-1} a_{\frac{mn}{d^2}}.$$

Since clearly $\alpha_0 = 0$ if $a_m = 0$, the Hecke operators map cusp forms to cusp forms.

Now let $\mathcal{H}_{\mathbb{C}} \subset \mathrm{End}(M_*)$ be the *Hecke algebra*, i.e. the $\mathbb{C}$-subalgebra generated by the Hecke operators. The key result is the following.

Theorem 2.5. *The action of $\mathcal{H}_{\mathbb{C}}$ on M_k is diagonalizable. Equivalently, there exists a basis of M_k consisting of simultaneous eigenforms of the $T(n)$.*

Proof. See [22], 2.B, Theorem 2. □

The proof of this theorem consists of two steps. First, one proves that the Hecke operators are self-adjoint with respect to the Petersson inner product:

$$\langle T(n)f, g \rangle = \langle f, T(n)g \rangle, \quad \text{for all } f, g \in M_k.$$

In particular, every $T(n)$ is diagonalizable. Secondly, one proves that the Hecke operators commute with each other, $T(m)T(n) = T(n)T(m)$, and now invokes the familiar result from linear algebra that a family of diagonalizable operators which pairwise commute is *simultaneously* diagonalizable.

The following proposition gives a more precise statement about the algebraic structure of the Hecke algebra.

[2]We should stress that this is not the natural definition of Hecke operators, for example it is not at all clear that $T(n)f$ is again a modular form of weight k. See for example [15, VII.5] for the "correct" definition.

Proposition 2.6. *The Hecke operators in weight k, $T(n) : M_k \to M_k$, satisfy the following relations:*

$$T(mn) = T(m)T(n), \qquad (m,n) = 1$$
$$T(p^{n+1}) = T(p)T(p^n) - p^{k-1}T(p^{n-1}), \quad n \geq 1,\ p \text{ prime.}$$

In particular, the Hecke algebra is generated by the set $\{T(p) \,|\, p \text{ prime}\}$.

Proof. See [15], VII.5.1, Proposition 10, or [22], 2.B, Theorem 1.(ii). □

Here are some important consequences of the results so far.

(i) For every k, the space M_k has a basis consisting of simultaneous eigenforms for the Hecke operators, called *Hecke eigenforms.* If $f = \sum_{m=0}^{\infty} a_m q^m$ is such an eigenform with eigenvalues $\{\lambda(n)\}_{n\geq 1}$, then

$$a_n = \lambda(n)a_1, \quad \text{for all } n.$$

It follows that the set of *normalized* Hecke eigenforms (i.e. eigenforms satisfying $a_1 = 1$) of weight k is a basis of M_k.

(ii) The Hecke eigenvalues $\{\lambda(n)\}$ of a given Hecke eigenform are real numbers, in fact real algebraic numbers (see [22], 2.B, Theorem 3), and satisfy the relations

$$\begin{aligned} \lambda(mn) &= \lambda(m)\lambda(n), & (m,n) = 1 \\ \lambda(p^{n+1}) &= \lambda(p)\lambda(p^n) - p^{k-1}\lambda(p^{n-1}), & n \geq 1,\ p \text{ prime.} \end{aligned} \tag{2}$$

In particular, by (i) above, the Fourier coefficients of a normalized Hecke eigenform are real algebraic numbers and satisfy the same recursive relations.

Example 2.7. Both G_k and Δ are normalized Hecke eigenforms of weights k and 12 respectively.

3. Periods of modular forms

In this section we introduce the periods of a modular form f and relate them to critical values of the L-series associated to f. In the case of Eisenstein series, the L-series in question is a product of Riemann zeta functions and we retrieve expressions for odd zeta values in terms of rapidly convergent Lambert series, first found by Ramanujan. The classical reference is [13], but our treatment is closer to [3].

3.1. **Differential forms associated to modular forms.** We begin by associating to a modular form f an $SL_2(\mathbb{Z})$-invariant differential form $\underline{f}$. To achieve this we need to introduce a suitable module of coefficients. Namely, for $k \geq 2$, let $V_k = \bigoplus_{0\leq n\leq k-2} \mathbb{Q}X^nY^{k-2-n}$ be the $\mathbb{Q}$-vector space of homogeneous polynomials of degree $k-2$. The group $SL_2(\mathbb{Z})$ acts on V_k on the right by

$$P(X,Y)|_\gamma := P(aX+bY, cX+dY), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ P(X,Y) \in V_k.$$

Definition 3.1. For a modular form $f \in M_k$, define

$$\underline{f}(z) = (2\pi i)^{k-1} f(z)(X - zY)^{k-2} \mathrm{d}z \in \Omega^1(\mathfrak{H}) \otimes V_k,$$

where $\Omega^1(\mathfrak{H})$ denotes the space of holomorphic one-forms on $\mathfrak{H}$.

Proposition 3.2. *We have $\underline{f}(\gamma.z)|_\gamma = \underline{f}(z)$, for all $\gamma \in SL_2(\mathbb{Z})$.*

Proof. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The following transformation formulas are easily verified,

$$(X - \gamma.zY)^{k-2}|\gamma = (cz+d)^{-k+2}(X - zY)^{k-2}, \quad \mathrm{d}\gamma.z = (cz+d)^{-2}\mathrm{d}z.$$

Using in addition that f is a modular form of weight k, we get

$$\begin{aligned} \underline{f}(\gamma.z)|\gamma &= (2\pi i)^{k-1} f(\gamma.z)(X - \gamma.zY)^{k-2}|\gamma \mathrm{d}\gamma.z \\ &= (2\pi i)^k \frac{(cz+d)^k}{(cz+d)^k} f(z)(X - zY)^{k-2}\mathrm{d}z \\ &= \underline{f}(z). \end{aligned}$$

Since $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ was arbitrary, the result follows. □

3.2. **Period polynomials of cusp forms.** Having constructed modular invariant differential one-forms associated to modular forms, we now define the analogues of $2\pi i$, restricting ourselves to cusp forms first.

Definition 3.3. Given a cusp form $f \in S_k$, define its *period polynomial* to be

$$P_f(X,Y) = \int_0^{i\infty} \underline{f}(z) \in V_k \otimes \mathbb{C}. \tag{3}$$

We also denote by

$$P_f^{\pm}(X,Y) := \frac{1}{2}(P_f(X,Y) \pm P_f(-X,Y))$$

the even/odd part with respect to the involution $X \mapsto -X$.

Convergence of the integral in (3) is ensured by $\underline{f}(z) = O(e^{2\pi i z})$ as $z \to i\infty$ and $\underline{f}(z) = O(e^{-2\pi i/z})$ as $z \to 0$ since f is a cusp form. Note that both conditions fail for the differential forms associated to Eisenstein series; in fact the corresponding integral (3) in fact diverges, but can be regularized (see Proposition 3.7).

Now expanding the term $(X - zY)^{k-2}$ in the integrand of (3), one gets

$$P_f(X,Y) = (2\pi i)^{k-1} \sum_{n=0}^{k-2} \left[(-1)^n \binom{k-2}{n} \int_0^{i\infty} f(z) z^n \mathrm{d}z \right] X^{k-2-n} Y^n,$$

and the complex number $p_n(f) := \int_0^{i\infty} f(z) z^n \mathrm{d}z$ is what's usually called in the literature the *n-th period* f.

Remark 3.4. The prefactor $(2\pi i)^{k-1}$ is omitted in many classical treatments of period polynomials and is included here following [3]. For the reader familiar with the general theory of periods, we mention that its purpose is to render the coefficients of $P_f(X,Y)$ *effective* periods, i.e. having non-negative weights, [12].

Example 3.5. For the cusp form Δ, we have

$$\begin{aligned} P_\Delta^+(X,Y) &= \omega_\Delta^+ \left(\frac{36}{691} X^{10} - X^8Y^2 + 3X^6Y^4 - 3X^4Y^6 + X^2Y^8 - \frac{36}{691} \right) \\ P_\Delta^-(X,Y) &= \omega_\Delta^- \left(4X^9Y - 25X^7Y^3 + 42X^5Y^5 - 25X^3Y^7 + 4XY^9 \right), \end{aligned}$$

for certain numbers $\omega_\Delta^+ \in \mathbb{R}$, $\omega_\Delta^- \in i\mathbb{R}$ which can in principle be computed explicitly to any given accuracy. Up to eleven digits of precision, we have, [6],

$$\omega_\Delta^+ = -68916772.809595194754\ldots, \quad \omega_\Delta^- = -i \times 5585015.3793104018668\ldots.$$

In general, Manin (see for example [13]) proved that the period polynomial of any cuspidal normalized Hecke eigenform has the form

$$P_f(X,Y) = \omega_f^+ Q_f^+(X,Y) + \omega_f^- Q_f^-(X,Y),$$

for some numbers $\omega_f^+ \in \mathbb{R}$, $\omega_f^- \in i\mathbb{R}$ and the polynomials $Q_f^\pm(X,Y)$ have coefficients in $\mathbb{Q}(f)$, the number field generated by the Hecke eigenvalues (=Fourier coefficients) of f. It is conjectured that the numbers $\omega_f^\pm$ are transcendental (in fact, algebraically independent over $\mathbb{Q}$) but this is not known in a single example.

3.3. **L-series of modular forms and the Mellin transform.** The periods $p_n(f)$ of the modular form f are closely related to special values of its L-series, defined as a formal series by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s \in \mathbb{C}$$

The growth properties of the coefficients a_n (see Proposition 2.2) imply that $L(f,s)$ converges absolutely and locally uniformly if $\mathrm{Re}(s) > k$ and even for $\mathrm{Re}(s) > \frac{k}{2}+1$ if f is cuspidal. Therefore, $L(f,s)$ defines an analytic function in some half plane of $\mathbb{C}$.

The relationship to periods is furnished by the following well-known formula which is an example of the *Mellin transform*.

Proposition 3.6. *For $f \in S_k$, we have*

$$\int_0^\infty f(it)t^{s-1}\,dt = (2\pi)^{-s}\Gamma(s)L(f,s),$$

for all s in the region of convergence of $L(f,s)$.

Proof. Replacing $f(z)$ by its Fourier expansion $\sum_{n=1}^\infty a_n q^n$ and interchanging summation and integration (which is okay since the series $\sum_{n=1}^\infty a_n e^{-2\pi nt}$ converges uniformly on $(0,\infty)$), we get

$$\begin{aligned}\int_0^\infty f(it)t^{s-1}\mathrm{d}t = \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi nt} t^{s-1}\mathrm{d}t &= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi nt}t^{s-1}\mathrm{d}t \\ &= (2\pi)^{-s}\sum_{n=1}^\infty \frac{a_n}{n^s}\int_0^\infty e^{-t}t^{s-1}\mathrm{d}t \\ &= (2\pi)^{-s}\Gamma(s)L(f,s).\end{aligned}$$

□

This important proposition has the following consequences.

(i) The function $s \mapsto L^*(f,s) := (2\pi)^{-s}\Gamma(s)L(f,s)$ has analytic continuation to the entire complex plane, and satisfies the functional equation $L^*(f,s) = (-1)^{\frac{k}{2}}L^*(f,k-s)$. The function $L^*(f,s)$ is called the *completed L-function* of f.

(ii) The periods of f are, up to a power of i, equal to the special values of $L^*(f,s)$ at the *critical points* $s = 1,\dots,k-1$, more precisely

$$p_n(f) = i^{n+1}L^*(f,n+1), \quad 0 \le n \le k-2.$$

So far, we have restricted our consideration to cusp forms. However, the case of Eisenstein series, while being technically slightly more involved, is also important. First of all, in that case the L-series in question is a product of Riemann zeta functions,

$$L(G_k,s)=\sum_{n=1}^{\infty}\frac{\sigma_{k-1}(n)}{n^s}=\zeta(s)\zeta(s-k+1).$$

Since $\zeta(s)$ has a simple pole at $s=1$, it is already clear that $L(f,s)$ has meromorphic rather than analytic continuation. However, exactly as before one proves that

$$\int_0^{\infty}(G_k(it)-a_0(G_k))t^{s-1}\mathrm{d}t=(2\pi)^{-s}\Gamma(s)L(G_k,s),\quad \text{if } \mathrm{Re}(s)>k,$$

where $a_0(G_k)=-\frac{B_k}{2k}$ is the constant term in the Fourier expansion of G_k. The difference to the case of cusp forms is that the integral on the left may diverge if $\mathrm{Re}(s)\leq k$ but can be regularized as follows. As before, we let $L^*(G_k,s):=(2\pi)^{-s}\Gamma(s)L(G_k,s)$.

Proposition 3.7. *For every $t_0>0$, we have*

$$L^*(G_k,s)=\int_{t_0}^{\infty}(G_k(it)-a_0(G_k))t^{s-1}dt+\int_0^{t_0}\left(G_k(it)-\frac{a_0(G_k)}{(it)^k}\right)t^{s-1}dt$$
$$-a_0(G_k)\left[\frac{t_0^s}{s}+\frac{i^k t_0^{s-k}}{k-s}\right],\quad \textit{if } \mathrm{Re}(s)>k.$$

Proof. A simple calculation using the fundamental theorem of calculus shows that d/dt_0 of the right hand side vanishes and therefore does not depend on t_0. But for $\mathrm{Re}(s)>k$, the limit as $t_0\to 0$ exists and equals $L^*(G_k,s)$ as we've seen above, and the result follows. ☐

The gain is now that the right hand side is well-defined for every $s\in\mathbb{C}$ (the second integral converges for every s because $G_k(it)-(it)^{-k}a_0(G_k)=O(e^{-2\pi/t})$, as $t\to 0$). In fact, setting $t_0=1$ and using modularity of G_k, we get

$$L^*(G_k,s)=\int_1^{\infty}(G_k(it)-a_0(G_k))(t^{s-1}+i^kt^{k-s-1})\mathrm{d}t-a_0(G_k)\left[\frac{1}{s}+\frac{i^k}{k-s}\right]. \tag{4}$$

Therefore, the function $s\mapsto L^*(f,s)$ has meromorphic continuation to the entire complex plane having simple poles at $s=0$ and $s=k$ with residues $-a_0(G_k)$ and $(-1)^{\frac{k}{2}}a_0(G_k)$ respectively, and one can verify that it also satisfies the functional equation $L^*(f,s)=(-1)^{k/2}L^*(f,k-s)$

A very interesting feature is that equation (4) gives rise to rapidly convergent formulas for odd zeta values $\zeta(k-1)$ which were first stated without proof by Ramanujan. We only give the result for $k=4$, the generalization to $k\equiv 0 \mod 4$ being straightforward while the case $k\equiv 2 \mod 4$ requires only slightly more work, [2].

Proposition 3.8. *We have*

$$\zeta(3)=\frac{7\pi^3}{180}-2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)}.$$

For the proof, we need the following lemma which is easily proved by differentiating both sides $k-1$ times with respect to t.

Lemma 3.9. *For every $k \geq 4$, we have*

$$\frac{1}{(2\pi)^{k-1}}\sum_{n=1}^{\infty}\sigma_{1-k}(n)e^{-2\pi nt} = \frac{1}{(k-2)!}\int_t^{\infty}(G_k(it')-a_0(G_k))(t'-t)^{k-2}dt',$$

where $\sigma_{1-k}(n) = \sum_{d|n}\frac{1}{d^{k-1}}$.

Proof of Proposition 3.8. Applying the preceding lemma for $k=4$ gives

$$\frac{2}{(2\pi)^3}\sum_{n=1}^{\infty}\sigma_{-3}(n)e^{-2\pi nt} = \int_t^{\infty}(G_4(it')-a_0(G_4))(t'-t)^2\mathrm{d}t'.$$

For $t=1$, the right hand side equals

$$\int_1^{\infty}(G_4(it')-a_0(G_4))(t'^2-2t'+1)\mathrm{d}t' = L^*(G_4,3)-L^*(G_4,2)+\frac{1}{3}a_0(G_4),$$

by (4), while the left hand side, again for $t=1$, equals

$$\frac{2}{(2\pi)^3}\sum_{m,n=1}^{\infty}\frac{e^{-2\pi mn}}{n^3} = \frac{2}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)}.$$

The statement now follows from the equalities

$$L^*(G_4,3) = -\frac{\zeta(3)}{(2\pi)^3},\quad L^*(G_4,2) = -\frac{1}{288},\quad \frac{a_0(G_4)}{3} = \frac{1}{720}.$$

□

3.4. **A digression on Euler products.** The L-series of modular forms should be thought of as relatives of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. A very important property of the latter is existence of an Euler product

$$\zeta(s) = \prod_p (1-p^{-s})^{-1},\quad \text{if } \mathrm{Re}(s) > 1.$$

where the product is over all prime numbers p. This formula is equivalent to a purely arithmetic statement, namely the unique factorization of positive integers into prime numbers.

Now for a general modular form, there is no reason to expect existence of a similar Euler product. In fact, it turns out that existence of an Euler product is deeply interrelated with the action of Hecke operators, as witnessed by the following result.

Proposition 3.10. *Let $f = \sum_{n=0}^{\infty} a_n q^n$ be a modular form of weight k. Then the associated L-series $L(f,s)$ has an Euler product*

$$L(f,s) = \prod_p (1-a_p p^{-s}+p^{k-1-s})^{-1}, \tag{5}$$

if and only if f is a normalized Hecke eigenform, (5) being valid for all s in the region of convergence, where again the product is taken over all prime numbers p.

Proof. See for example [8], Theorem 5.9.2. □

The point is that by expanding all factors in the Euler product as geometric series, one obtains precisely the recurrence relations satisfied by the coefficients of normalized Hecke eigenforms, (2).

An important consequence is that L-series of normalized Hecke eigenforms are nonzero in the region of convergence. In particular $L(f,s) \neq 0$ for all $\mathrm{Re}(s) > (k+1)/2$, and by the functional equation, likewise for $\mathrm{Re}(s) < (k-1)/2$, except for "trivial zeros" at negative integers $s \in \{-1,-2,\ldots\}$ which come from the poles of $\Gamma(s)$. In particular, the periods $p_0(f), p_1(f), \ldots, p_{k-2}(f)$ of a Hecke eigencusp form f of weight k are all nonzero except for possibly $p_{k/2-1}(f)$ which corresponds to the "central L-value" $L(f,k/2)$.

Remark 3.11. The region $(k-1)/2 \leq \mathrm{Re}(s) \leq (k+1)/2$ for which functional equation together with Euler product does not make any predictions about nonvanishing is called the *critical strip*, in analogy with the critical line $\mathrm{Re}(s) = 1/2$ of the Riemann zeta function. Indeed, note that $L(f,k/2) = 0$ if $k \equiv 2 \mod 4$ since the sign $(-1)^{k/2}$ in the functional equation is then odd.

3.5. **Why the name "periods of modular forms"?** We now explain why the periods of modular forms can be considered analogues of $2\pi i$.

Recall that by virtue of Cauchy's integral formula the number $2\pi i$ arises as the monodromy-period of the logarithm:

$$2\pi i = \int_{\sigma_0} \frac{dz}{z},$$

where σ_0 is a loop winding once around zero in the positive direction. Choosing $1 \in \mathbb{C}^\times$ as our base point and pulling the integral back along the universal cover $\varphi : \mathbb{C} \to \mathbb{C}^\times$, $\xi \mapsto e^\xi$, this is equivalent to

$$2\pi i = \int_0^{2\pi i} \mathrm{d}\xi,$$

because $\mathrm{d}\xi = \varphi^*\left(\frac{\mathrm{d}z}{z}\right)$. In other words, $2\pi i$ is equal to an integral of the natural one-form $\mathrm{d}\xi$ on $\mathbb{C}$ between two points in the fiber $\varphi^{-1}(1)$. Moreover, for any $a, b \in \varphi^{-1}(1)$, we have $\int_a^b \mathrm{d}\xi \in 2\pi i\mathbb{Z}$.

Now the modular analogue of the universal cover $\mathbb{C} \to \mathbb{C}^\times$ is the natural projection

$$\psi : \mathfrak{H}^* \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}^*,$$

where $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1_{\mathbb{Q}}$ is the extended upper half-plane obtained by adding all rational points on the real line including the point "at infinity" $i\infty$ (or $[1:0]$ in projective coordinates), and the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1_{\mathbb{Q}}$ is given by $\gamma.[p_0 : p_1] = [ap_0 + bp_1 : cp_0 + dp_1]$, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Note that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{P}^1_{\mathbb{Q}} \cong \{[1:0]\}$ and therefore $\psi^{-1}(i\infty) = \mathbb{P}^1_{\mathbb{Q}}$.

By definition the period polynomial of f is obtained by integrating the $\mathrm{SL}_2(\mathbb{Z})$-invariant differential $\underline{f}$ between the points 0 and $i\infty$,

$$\int_0^{i\infty} \underline{f} = \int_{S^{-1}.i\infty}^{i\infty} \underline{f}.$$

More generally, one can show that for any $Q_1, Q_2 \in \mathbb{P}^1_{\mathbb{Q}}$, there exists $\gamma \in \mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$ (the group ring), such that

$$\int_{Q_1}^{Q_2} \underline{f} = \underbrace{\left(\int_0^{i\infty} \underline{f}\right)}_{P_f(X,Y)}\Bigg|\gamma,$$

where the first integral is taken along a geodesic between Q_1 and Q_2 in $\mathfrak{H}^*$. Therefore, the coefficients of $\int_{Q_1}^{Q_2} \underline{f}$ are $\mathbb{Z}$-linear combinations of the coefficients of $P_f(X,Y)$. Rather than proving this last fact, which is a well-known result in the theory of so-called modular symbols, [13], we give an example which will make the general pattern clear.

Example 3.12. Assume that $Q_1 = [3:5]$, $Q_2 = [2:5]$ which correspond to the points $3/5, 2/5 \in \mathbb{Q}$. For any cusp form f, we have

$$\int_{3/5}^{2/5} \underline{f} = \left(\int_{5/3}^{5/2} \underline{f}\right)\Bigg|_{S^{-1}},$$

since $S.(3/5) = 5/3$, $S.(2/5) = 5/2$ and $\underline{f}$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant. The same argument applied to $T^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ gives

$$\int_{5/3}^{5/2} \underline{f} = \left(\int_{-1/3}^{1/2} \underline{f}\right)\Bigg)\Bigg|T^{-2}.$$

Now we simply split the integral

$$\int_{-1/3}^{1/2} \underline{f} = \int_{-1/3}^{0} \underline{f} + \int_0^{1/2} \underline{f},$$

and apply the preceding argument now again with S in both integrals separately. We get

$$\int_{-1/3}^{0} \underline{f} + \int_0^{1/2} \underline{f} = \left(\int_3^{i\infty} \underline{f}\right)\Bigg|_{S^{-1}} + \left(\int_{i\infty}^{-2} \underline{f}\right)\Bigg|_{S^{-1}}.$$

Finally, we apply the same argument as before with T_3 in the first integrand and with T^{-2} in the second, also reverting the integration domain, to obtain

$$\int_3^{i\infty} \underline{f} + \int_{i\infty}^{-2} \underline{f} = \left(\int_0^{i\infty} \underline{f}\right)\Bigg|_{T^{-3}} - \left(\int_0^{i\infty} \underline{f}\right)\Bigg|_{T^2}.$$

Putting together everything we've done so far, we see that

$$\int_{3/5}^{2/5} \underline{f} = \left(\int_0^{i\infty} \underline{f}\right)\Bigg|_{\gamma},$$

with $\gamma = (T^{-3} - T^{-2})S^{-1}T^{-2}S^{-1} \in \mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$.

4. The Eichler–Shimura theorem

In the preceding section we introduced the period polynomial $P_f(X,Y) \in V_k$ of a cusp form f and showed that its coefficients are, up to elementary factors, the values of the completed L-function $L(f,s)$ of f at the "critical points" $s \in \{1,\dots,k-1\}$ where k is the weight of f.

In this section we will see that period polynomials satisfy two functional equations, the *period relations*, which reflect their origin as integrals of modular forms. As a second result, we will see that the subspace of V_k consisting of period polynomials can be characterized using the period relations.

4.1. **The period relations.** We being by remarking that the group $\mathrm{SL}_2(\mathbb{Z})$, which we already know is generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, is also generated by S and $U = TS = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. As $S^2 = U^3 = -I$, ($I = 2 \times 2$ identity matrix) this shows that $\mathrm{SL}_2(\mathbb{Z})$ is in fact generated by two elements of finite order.

Now let f be a cusp form with associated period polynomial $P_f(X,Y)$.

Proposition 4.1. *The polynomial $P_f(X,Y)$ satisfies the period relations*

$$P_f(X,Y) + P_f(-Y,X) = 0,$$
$$P_f(X,Y) + P_f(X-Y,X) + P_f(-Y,X-Y) = 0.$$

Equivalently,

$$P_f(X,Y)|(1+S) = 0,$$
$$P_f(X,Y)|(1+U+U^2) = 0,$$

where the right-action $(\gamma, P) \mapsto P(X,Y)|\gamma$ has been extended linearly to the group ring $\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$.

Proof. Using invariance $\underline{f}(\gamma.z)|_\gamma = \underline{f}$ and composition of paths, we have

$$\begin{aligned} P_f(X,Y)|(1+S) &= \int_0^{i\infty} \underline{f} + \left(\int_0^{i\infty} \underline{f}\right)|S = \left(\int_0^{i\infty} + \int_{S^{-1}.0}^{S^{-1}.i\infty}\right)\underline{f} \\ &= \left(\int_0^{i\infty} + \int_{i\infty}^{0}\right)\underline{f} \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} P_f(X,Y)|(1+U+U^2) &= \int_0^{i\infty} \underline{f} + \left(\int_0^{i\infty} \underline{f}\right)|U + \left(\int_0^{i\infty} \underline{f}\right)|U^2 \\ &= \left(\int_0^{i\infty} + \int_{U^{-1}.0}^{U^{-1}.i\infty} + \int_{U^{-2}.0}^{U^{-2}.i\infty}\right)\underline{f} \\ &= \left(\int_0^{i\infty} + \int_1^{0} + \int_1^{\infty}\right)\underline{f} \\ &= 0. \end{aligned}$$

□

4.2. **The Eichler–Shimura isomorphism.** Now consider for even $k \geq 2$ the subspace

$$W_k = \{P(X,Y) \in V_k \mid P(X,Y)|(1+S) = P_f(X,Y)|(1+U+U^2) = 0\}$$

of polynomials satisfying the period relations. This space is stable under the action of the element $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ by conjugation, and therefore we may decompose W_k into its ± 1-eigenspace under this action

$$W_k = W_k^+ \oplus W_k^-.$$

This amounts to saying that if $P(X,Y) \in V_k$ satisfies the period relations, then so do the two polynomials $P^{\pm}(X,Y) = \frac{1}{2}(P(X,Y) \pm P(X,-Y)) \in V_k^{\pm}$.

Theorem 4.2 (Eichler–Shimura)**.** *The period maps*

$$\psi^{\pm} : S_k \to W_k^{\pm} \otimes \mathbb{C}, \quad f \mapsto P_f^{\pm}(X,Y)$$

are both injective. Moreover, ψ^- induces a $\mathbb{C}$-linear isomorphism $S_k \cong W_k^- \otimes \mathbb{C}$ while $\mathrm{Im}(\psi^+) \subset W_k^+ \otimes \mathbb{C}$ is a codimension one subspace not containing the element $X^{k-2} - Y^{k-2}$.

Sketch of proof. Given two cusp forms $f, g \in S_k$, it is a result of Haberland (see [22]) that

$$\langle f, g \rangle = \sum_{\substack{m+n \leq k-2 \\ m \not\equiv n \mod 2}} \frac{(k-2)!}{m!n!(k-2-m-n)!} p_m(f) p_n(g).$$

Therefore, if all even (respectively all odd) periods of f vanish, then $\langle f, f \rangle = 0$, hence $f = 0$ since the Petersson inner product is non-degenerate. This shows that the maps $\psi^{\pm}$ are both injective.

The fact that ψ^- is an isomorphism onto W_k^- and that $\mathrm{Im}(\psi^+)$ has codimension one in W_k^+ are equivalent to the dimension formulas

$$\dim W_k^+ = \dim S_k + 1 = \dim M_k \quad \dim W_k^- = \dim S_k,$$

which can be proved using invariant theory. For details, see [21]. □

The upshot of the preceding theorem is that firstly, every cusp form f is uniquely determined by either its even periods $p_0(f), p_2(f), \ldots, p_{k-2}(f)$ or its odd periods $p_1(f), p_3(f), \ldots, p_{k-3}(f)$. Secondly, at least after tensoring with $\mathbb{C}$, the space of period polynomials admits a purely combinatorial description of solutions to the period relations (there is one extra linear equation which characterizes $\mathrm{Im}(\psi^+)$ inside of W_k^+, see [11]).

5. The Rankin–Selberg method

In this section we will relate periods of a Hecke eigencusp form to the Petersson inner product of f with a product of Eisenstein series which uses a technique introduced independently by Rankin and Selberg. The core of the method is the characteristic "unfolding trick" which reduces an integral over a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ to a simpler one.

We then give a concrete application to demonstrate that the formalism of periods of modular forms developed so far is useful even if one is only interested in modular forms.

5.1. **Real analytic Eisenstein series as integration kernels.** In this section, we denote $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\infty = \langle -I, T\rangle \subset \Gamma$. Then Γ_∞ is precisely the stabilizer of the cusp $i\infty$. Further we denote by $\Gamma_\infty\backslash\Gamma$ the set of right cosets of Γ_∞ in Γ.

Definition 5.1. For a non-negative integer $k \geq 0$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -k/2+1$, define the *real analytic Eisenstein series* of weight k to be

$$\mathcal{G}_k^s(z) := \sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \frac{y^s}{(cz+d)^k|cz+d|^{2s}}.$$

The assumption on s ensures absolute and locally uniform convergence of the above series, and therefore $\mathcal{G}_k^s(z)$ defines a real analytic function of z.

For $k \geq 4$, the real analytic Eisenstein series $\mathcal{G}_k^s(z)$ is related to the usual Eisenstein series $G_k(z)$ by

$$\mathcal{G}_k^0(z) = \frac{1}{2\zeta(k)}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(mz+n)^k} = -\frac{2k}{B_k}G_k(z).$$

Also, one verifies that for every s as above, the function $\mathcal{G}_k^s$ transforms like a modular form of weight k:

$$\mathcal{G}_k^s(\gamma.z) = (cz+d)^k\mathcal{G}_k(z), \quad \text{for all } \gamma\in\Gamma.$$

One of the main reasons for considering real analytic Eisenstein series is that they define suitable integration kernels (via the Petersson inner product) which can be used to extract arithmetic information out of modular forms. The basic result is the following.

Theorem 5.2 (Rankin–Selberg)**.** *Let h, h' be positive even integers, and $k = h+h'$. Given $f = \sum_{n=1}^\infty a_nq^n \in S_k$ and $g = \sum_{n=0}^\infty b_nq^n \in M_{h'}$, we have*

$$\langle f, \mathcal{G}_h^s\cdot g\rangle = \frac{\Gamma(k-1+s)}{(4\pi)^{k-1+s}}\sum_{n=1}^\infty \frac{a_n\overline{b_n}}{n^{k-1+s}},$$

for $\mathrm{Re}(s) > h'+1-k/2$.

The series on the right is the *convolution L-series* of f and g and is sometimes denoted $L(f\times g, s')$ (here $s' = k-1+s$).

Proof. By definition, we have (with $\mathrm{d}\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$)

$$\langle f, \mathcal{G}_h^s\cdot g\rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{\mathcal{G}_h^s(z)g(z)}y^k\mathrm{d}\mu = \int_{\Gamma\backslash\mathfrak{H}}\sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma}\frac{f(z)\overline{g(z)}}{(c\overline{z}+d)^h|cz+d|^{2s}}y^k\mathrm{d}\mu.$$

Using the modular transformation properties of f, g and $y = \mathrm{Im}(z)$, as well as invariance of $\mathrm{d}\mu$, the last line equals

$$\int_{\Gamma\backslash\mathfrak{H}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma.z)\overline{g(\gamma.z)}\,\mathrm{Im}(\gamma.z)^{k+s}\gamma^*\mathrm{d}\mu.$$

Now since $\Gamma_\infty\backslash\mathfrak{H} = \dot{\bigcup}_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma(\Gamma\backslash\mathfrak{H})$ (union of disjoint sets up to sets of measure zero), and since $(0,1)\times(0,\infty)$ is a fundamental domain for the action of Γ_∞ on $\mathfrak{H}$, we can rewrite the last integral as

$$\int_{\Gamma_\infty\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k+s}\mathrm{d}\mu = \int_0^\infty \int_0^1 f(z)\overline{g(z)}y^{k+s-2}\mathrm{d}x\mathrm{d}y$$

$$\int_0^\infty \left(\int_0^1 \sum_{\substack{m\geq 1\\ n\geq 0}} a_m e^{2\pi i m(x+iy)}\overline{b}_n \overline{e^{2\pi i n(x+iy)}}\mathrm{d}x \right) y^{k-2+s}\mathrm{d}y$$

$$= \int_0^\infty \sum_{n=1}^\infty a_n\overline{b}_n e^{-4\pi n y}y^{k-2+s}\mathrm{d}y$$

$$= \frac{\Gamma(k-1+s)}{(4\pi)^{k-1+s}}\sum_{n=1}^\infty \frac{a_n\overline{b}_n}{n^{k-1+s}},$$

where we used orthonormality $\int_0^1 e^{2\pi i(m-n)x}\mathrm{d}x = \delta_{m,n}$, and in the last equality we interchanged integration and summation, which is okay since the sum in the integrand converges uniformly in y. □

5.2. **Consequences.** Some special cases of the Rankin–Selberg theorem are of particular interest. Here is a first one.

Theorem 5.3 (Addendum to Rankin–Selberg)**.** *With notation as in the previous theorem, assume in addition that $f \in S_k$ is a normalized Hecke eigencusp form and that $g = G_{h'}$. Then*

$$\langle f, \mathcal{G}_h^s \cdot G_{h'}\rangle = \frac{\Gamma(k-1+s)}{(4\pi)^{k-1+s}\zeta(h+2s)}L(f,h+s)L(f,k-1+s),$$

for all s as before.

Sketch of proof. This is more or less purely combinatorial, and can be deduced from the following facts:

(i) Since $f = \sum_{n=0}^\infty a_n q^n$ is a normalized Hecke eigenform, its L-series has an Euler product which looks as follows:

$$L(f,s) = \prod_p \frac{1}{1-a_p p^{-s}+p^{k-1-2s}} = \prod_p \frac{1}{(1-\alpha_p^1 p^{-s})(1-\alpha_p^2 p^{-s})},$$

for some $\alpha_p^1, \alpha_p^2 \in \mathbb{C}$, where the product is over all primes.

(ii) If $g = \sum_{n=0}^\infty b_n q^n \in M_{h'}$ is another normalized Hecke eigenform with Euler product

$$L(g,s) = \prod_p \frac{1}{(1-\beta_p^1 p^{-s})(1-\beta_p^2 p^{-s})},$$

then the convolution L-series $L(f\times g, s') := \sum_{n=1}^\infty \frac{a_n b_n}{n^{s'}}$ has an Euler product given by

$$\zeta(2(s'+1)-k-h')^{-1}\prod_p \prod_{1\leq i,j\leq 2}\frac{1}{(1-\alpha_p^i\alpha_p^j p^{-s'})}.$$

For a proof, see [7], Theorem 1.6.3.

(iii) The Euler product of $L(G_{h'},s)=\zeta(s)\zeta(s-h'+1)$ is given by
$$\prod_p \frac{1}{(1-p^{-s})(1-p^{h'-1-s})},$$
i.e. $\alpha_p^1=1$, $\alpha_p^2=p^{h'-1}$.

Using these facts, we get from Theorem 5.2 for $g=G_{h'}$
$$\begin{aligned}\langle f,\mathcal{G}_h^s G_{h'}\rangle &= \frac{\Gamma(s')}{(4\pi)^{s'}}\sum_{n=1}^{\infty}\frac{a_n\bar{b}_n}{n^{s'}}\\ &= \zeta(2(s'+1)-k-h')^{-1}\prod_p\prod_{1\leq i,j\leq 2}\frac{1}{(1-\alpha_p^i\alpha_p^j p^{-s'})}\\ &= \zeta(2(s'+1)-k-h')^{-1}L(f,s')L(f,s'-h'+1)\end{aligned}$$
where we also used that the Fourier coefficients of $G_{h'}$ are rational numbers, in particular real. Now setting $s'=k-1+s$ gives the result. □

Corollary 5.4. *For $h,h'\geq 4$ even, $f\in S_k$ a normalized Hecke eigencusp form, we have*
$$\langle f,G_hG_{h'}\rangle=\frac{p_{h-1}(f)p_{k-2}(f)}{(2i)^{k-1}}.$$

Proof. This follows from setting $s=0$ in Theorem 5.3 and using the formulas
$$p_n(f)=\frac{i^{n+1}n!}{(2\pi)^{n+1}}L(f,n+1),\quad \zeta(h)=-\frac{B_h(2\pi i)^h}{2h!}\ (h\geq 2\text{ even})$$
and $\mathcal{G}_h^0(z)=-\frac{2h}{B_h}G_h(z)$. □

This formula can be extended with slight extra care also to $h=2$. The problem is that G_2 is not a modular form, however, one can verify that for $k\geq 6$ the linear combination
$$G_2(z)G_{k-2}(z)+\frac{G'_{k-2}(z)}{4\pi i(k-2)},$$
called the *Serre derivative* of G_{k-2}, is a modular form of weight k. With this, one can show that
$$\left\langle f,G_2G_{k-2}+\frac{G'_{k-2}}{4\pi i(k-2)}\right\rangle=\frac{p_1(f)p_{k-2}(f)}{(2i)^{k-1}},$$
so that Corollary 5.4 extends to $h=2$ also.

5.3. **Petersson inner product for all modular forms.** As a final application of the Rankin–Selberg method, we describe a natural extension of the Petersson inner product to all modular forms following [19]. For this, we need that the real analytic Eisenstein series of weight zero $\mathcal{G}_0^s(z)$ has meromorphic continuation to all $s\in\mathbb{C}$ with $\mathrm{Res}_{s=1}\mathcal{G}_0^s(z)=\frac{3}{\pi}$ for all $z\in\mathfrak{H}$, [7]. Combined with the Rankin–Selberg theorem, we get
$$\langle f,g\rangle=\frac{\pi}{3}\mathrm{Res}_{s=1}\langle f,\mathcal{G}_0^s g\rangle=\frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}L(f\times g,s).$$

for all $f \in S_k$ and $g \in M_k$. In particular, for f a normalized Hecke eigencusp form and $g = G_k$, we get from Theorem 5.3

$$\langle f, g\rangle = \frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\operatorname{Res}_{s=k}\left(\frac{L(f,s)L(f,s-k+1)}{\zeta(2(s-1+k))}\right).$$

Firstly this shows again that $\langle f, G_k\rangle = 0$ for every (not necessarily normalized) Hecke eigencusp form f since $L(f,s)$ has analytic continuation. More importantly, the right hand side makes sense for $f = G_k$ also, and using that $L(G_k, s) = \zeta(s)\zeta(s-k+1)$, we may define

$$\langle G_k, G_k\rangle := \frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\operatorname{Res}_{s=k}\left(\frac{\zeta(s)\zeta(s-k+1)^2\zeta(s-2k+2)}{\zeta(2(s-k+1))}\right).$$

It is known that $\zeta(s)$ has simple pole at $s = 1$ with residue one, a simple zero at $s = 2-k$ for $k > 4$ and that $\zeta'(2-k) = \frac{\pi i(k-2)!}{(2\pi i)^{k-1}}\zeta(k-1)$. Using this, we get after a short calculation

$$\langle G_k, G_k\rangle = \frac{(k-2)!}{(4\pi)^{k-1}}\frac{B_k}{2k}\zeta(k-1).$$

Note that this is positive if and only if $k \equiv 2 \mod 4$ so that the natural extension of $\langle\cdot,\cdot\rangle$ to all of M_k is in general not positive definite.

5.4. **A theorem of Rankin–Zagier.** In this section we prove a result about the structure of M_k, the space of modular forms of weight k which could have already been stated in Section 1. Its proof, however, will use much of the theory we set up so far, including injectivity of the period map (a part of the Eichler–Shimura theorem) as well as the Rankin–Selberg method. To the best of the author's knowledge, no proof is known which avoids these two results.

Theorem 5.5. *The space M_k is $\mathbb{C}$-linearly spanned by the set*

$$U_k := \{G_k, [G_h, G_{h'}] \mid h + h' = k\},$$

where $[G_h, G_{h'}](z) = G_h(z)G_{h'}(z) + \frac{1}{4\pi i}\left(\frac{\delta_{h,2}G'_{h'}(z)}{h'} + \frac{\delta_{h',2}G'_h(z)}{h}\right)$.

The expression $[G_h, G_{h'}]$, which for $h, h' \geq 4$ is simply the product of G_h and $G_{h'}$, is a modular form of weight k also known as the (0-th) Rankin–Cohen bracket of G_h and $G_{h'}$ (see for example [23], Section 5.2).

Remark 5.6. It is instructive to compare this theorem with the statement

$$M_k = \operatorname{Span}_{\mathbb{C}}\{G_4^a G_6^b \mid 4a + 6b = k\},$$

which is a consequence of Theorem 2.3.(ii). In particular, a simple counting argument shows that U_k cannot be a basis. On the other hand, a certain subset of U_k is a basis of M_k, [10].

Proof. For $k < 12$, we have $\dim M_k \leq 1$, so that U_k generates M_k in these cases. We may therefore assume that $k \geq 12$.

Let $g \in U_k^{\perp} = \{f \in M_k \mid \langle f, h\rangle = 0, \forall h \in U_k\}$ be an element of the orthogonal complement of U_k in M_k, which we may write uniquely as $g = \widetilde{g} + \lambda G_k$, with $\widetilde{g} \in S_k$, and $\lambda \in \mathbb{C}$. Since $G_k \in U_k$ which is orthogonal to all cusp forms (see Section 5.4), we have $\lambda = 0$ and therefore $g = \widetilde{g}$ is cuspidal.

Now consider the $\mathbb{C}$-linear map

$$\psi : S_k \to V_k^-$$
$$f \mapsto \sum_{\substack{m,n\geq 2\\ m+n=k}} (-1)^{n-1}\binom{k-2}{n-1}\langle f,[G_m,G_n]\rangle X^{m-1}Y^{n-1}.$$

By Rankin–Selberg (Corollary 5.4), for every Hecke-normalized eigenform $f \in S_k$, we have

$$\psi(f) = \frac{p_{k-2}(f)}{(-4\pi)^{k-1}}P_f(X,Y)^-.$$

Writing $g = \sum_i \lambda_i f_i$ as a $\mathbb{C}$-linear combination of Hecke-normalized eigenforms f_i, we therefore have

$$0 = \psi(g) = \sum_i \lambda_i \psi(f_i) = \sum_i c_i P_{f_i}(X,Y)^-, \quad c_i := \frac{\lambda_i p_{k-2}(f_i)}{(-4\pi)^{k-1}}.$$

By Eichler–Shimura (Theorem 4.2), the $P_{f_i}(X,Y)^-$ are linearly independent, so that $c_i = 0$ for all i. On the other hand, we have

$$p_{k-2}(f_i) = \frac{i^{k-1}(k-2)!}{(2\pi)^{k-1}}L(f_i,k-1) \neq 0,$$

since $L(f_i,k-1)$ has an Euler product expansion, all of whose factors are non-zero. Therefore we must have $\lambda_i = 0$ for all i, that is $g = 0$. $\square$

6. Modular invariant primitives of modular forms

In this section we construct analogues of the single-valued logarithm $z \mapsto \log|z|$ for modular forms. For Eisenstein series this is done in detail and leads to the real analytic Eisenstein series already encountered in conjunction with the Rankin–Selberg method. On the other hand, the analogous construction for cusp forms is significantly harder and is only indicated in the final subsection.

6.1. **The logarithm, again.** To motivate the contents of this section we again return to our basic example the logarithm. For a point $z \in \mathbb{C}^\times$, let $\gamma_z : [0,1] \to \mathbb{C}^\times$ be a path from 0 to z, i.e. $\gamma_z(0) = 1$ and $\gamma_z(1) = z$, and consider the line integral $\int_{\gamma_z} \frac{\mathrm{d}x}{x}$. Now if $\widetilde{\gamma}_z$ is another such path, then

$$\left(\int_{\widetilde{\gamma}_z} - \int_{\gamma_z}\right)\frac{\mathrm{d}x}{x} \in 2\pi i\mathbb{Z}.$$

Indeed, on the universal cover $\pi : \mathbb{C} \to \mathbb{C}^\times$, $\xi \mapsto z = e^\xi$ the preceding integral pulls back to

$$\left(\int_{\gamma_z} - \int_{\widetilde{\gamma}_z}\right)\frac{\mathrm{d}x}{x} = \int_0^{2\pi i n}\mathrm{d}\xi = 2\pi i n,$$

where the integer $n \in \mathbb{Z}$ (the *winding number*) counts the number of times the loop $\widetilde{\gamma}_z^{-1} \circ \gamma_z$ goes around 0. In other words for fixed z the value $\int_{\gamma_z}\frac{\mathrm{d}x}{x}$ depends on the choice of path γ_z up to addition of an integer multiple of $2\pi i$.

The point is now that since $2\pi i$ is a purely imaginary number, the real part of the integral $\int_{\gamma_z} \frac{\mathrm{d}x}{x}$ does in fact only depend on z and not on the path, and we have

$$\mathrm{Re}\left(\int_{\gamma_z} \frac{\mathrm{d}x}{x}\right) = \log|z|.$$

Another way of characterizing the function $\mathbb{C}^\times \to \mathbb{R}$, $z \mapsto \log|z|$ thus obtained is as the unique solution to the differential equation

$$\mathrm{d}F = \mathrm{Re}\left(\frac{\mathrm{d}z}{z}\right) = \frac{1}{2}\left(\frac{\mathrm{d}z}{z} + \frac{\mathrm{d}\overline{z}}{\overline{z}}\right)$$

such that $F(1) = 0$.

6.2. **Monodromy of integrals of modular forms.** We would like to mimick the construction of the preceding section for integrals of modular forms, the latter being defined as follows.

Definition 6.1. Given a modular form $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ and a point $z \in \mathfrak{H}$ define its indefinite integral to be

$$I_f(z) = \int_z^{i\infty} \underline{f}^0 - \int_0^z \underline{f}^\infty,$$

where

$$\underline{f}^\infty := (2\pi i)^{k-1} a_0 (X - zY)^{k-2} \mathrm{d}z, \quad \underline{f}^0 := \underline{f} - \underline{f}^\infty.$$

It is helpful to think of $\underline{f}^\infty$ as the constant term of $\underline{f}$. The point $i\infty$ plays here the role of a fixed canonical base point, analogous to $1 \in \mathbb{C}^\times$ in the case of the logarithm.

Remark 6.2. In the definition above we have been slightly sloppy; the integrals are defined as path integrals so we really should have chosen paths from z to $i\infty$ and from 0 to z. However, it turns out that the above integrals are independent of the choice of path, so we don't need to specify them.

Clearly the function $\mathfrak{H} \to V_k \otimes \mathbb{C}$, $z \mapsto I_f(z)$ is a primitive of $-\underline{f}$ in the sense that $\frac{\partial}{\partial z} I_f(z) = -\underline{f}$.

Definition 6.3. We say that a function $F : \mathfrak{H} \to V_k \otimes \mathbb{C}$ is *modular invariant* if $F(\gamma.z)|_\gamma = F(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

It turns out that the $I_f(z)$ are not modular invariant. Indeed, for cusp forms, we already know this since the following proposition will show that the defect is precisely measured by the period polynomial of f and the latter is always nonzero by the Eichler–Shimura theorem.

Proposition 6.4. *If $f \in S_k$ is a cusp form, then*

$$I_f(T.z)|T = I_f(z), \quad I_f(z) - I_f(S.z)|S = P_f(X, Y),$$

where $P_f(X, Y)$ is the period polynomial of f.

Proof. Since f is a cusp form, we have $\underline{f}^0 = \underline{f}$, hence

$$I_f(z) = \int_z^{i\infty} \underline{f}.$$

To see invariance under T, note that $T^{-1}.i\infty = i\infty$ and therefore

$$\int_{T.z}^{i\infty} \underline{f}|T = \int_{z}^{T^{-1}.i\infty} \underline{f}(T.z)|T = \int_{z}^{i\infty} \underline{f},$$

since $\underline{f}$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, by Proposition 3.2.

For the second statement, using again $\mathrm{SL}_2(\mathbb{Z})$-invariance of the integrand and the fact that $S^{-1}.i\infty = 0$, we get

$$I_f(z) - I_f(S.z)|S = \int_z^{i\infty} \underline{f} - \int_z^0 \underline{f} = \int_0^{i\infty} \underline{f},$$

which is precisely the definition of $P_f(X,Y)$. □

Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, the preceding proposition completely determines the monodromy of the integrals $I_f(z)$.

Similarly, we now compute the monodromy of integrals of Eisenstein series of weight $2k \geq 4$.

Proposition 6.5 (Haberland, [9], 1.4.)**.** *We have*

$$I_{G_{2k}}(T.z)|T - I_{G_{2k}}(z) = \frac{(2\pi i)^{2k-1}}{2k-1}\frac{B_{2k}}{4k}\frac{(X+Y)^{2k-1} - X^{2k-1}}{Y},$$

$$I_{G_{2k}}(S.z)|S - I_{G_{2k}}(z) = \frac{(2k-2)!}{2}\Bigg(\zeta(2k-1)(Y^{2k-2} - X^{2k-2}) + (2\pi i)^{2k-1}\sum_{n=1}^{k-1}\frac{B_{2n}B_{2k-2n}}{(2n)!(2k-2n)!}X^{2n-1}Y^{2k-2n-1}\Bigg).$$

Proof. We begin with the transformation under T. By definition, we have

$$\begin{aligned} I_{G_{2k}}(T.z)|T - I_{G_{2k}}(z) &= \int_{T.z}^{i\infty} \underline{G}_{2k}^0|T - \int_z^{i\infty} \underline{G}_{2k}^0 - \int_0^{T.z} \underline{G}_{2k}^\infty|T + \int_0^z \underline{G}_{2k}^\infty \\ &= -\int_{T^{-1}.0}^{z} \underline{G}_{2k}^\infty + \int_0^z \underline{G}_{2k}^\infty \\ &= -\int_{-1}^{0} \underline{G}_{2k}^\infty \\ &= -(2\pi i)^{2k-1} a_0(G_{2k}) \int_0^1 (X+zY)^{2k-2}\mathrm{d}z. \end{aligned}$$

Computation of the final integral is straightforward and the result follows upon noting that $a_0(G_{2k}) = -\frac{B_{2k}}{4k}$.

For the transformation under S, a straightforward computation using equation 4 yields for $z = i$

$$I_{G_{2k}}(i)|S - I_{G_{2k}}(i) = (2\pi i)^{2k-1}\sum_{r=0}^{2k-2}\binom{2k-2}{r}(-i)^{r+1}L^*(G_{2k}, r+1)X^{2k-2-r}Y^r,$$

where we also used that $S.i = i$. Now since $L^*(G_{2k}, s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-2k+1)$, using the well-known formulas $\zeta(0) = -\frac{1}{2}$, $\zeta(2k) = -\frac{B_{2k}(2\pi i)^{2k}}{2(2k)!}$, $\zeta(1-2k) = -\frac{B_{2k}}{2k}$ as well as $\zeta'(-2k) = \frac{(2k)!}{2}\frac{\zeta(2k+1)}{(2\pi i)^{2k}}$, valid for all $k \geq 1$, we get the result.

□

6.3. **Modular invariant primitives of Eisenstein series.** Having computed the monodromy of the functions $I_f(z)$, we would somehow like to eliminate it to construct modular invariant primitives. If f is an Eisenstein series, this can be done as follows.

Definition 6.6. For $k \geq 1$, define

$$\mathcal{E}_{2k}(z) := (2\pi i)^{-2k}\,\mathrm{Re}\left(I_{G_{2k+2}}(z) + \frac{(2k)!}{2}\zeta(2k+1)Y^{2k}\right).$$

Proposition 6.7. *The function $\mathcal{E}_{2k}(z)$ is modular invariant,*

$$\mathcal{E}_{2k}(\gamma.z)|\gamma = \mathcal{E}_{2k}(z), \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}), \tag{6}$$

and satisfies the differential equation

$$d\mathcal{E}_{2k}(z) = \mathrm{Re}(2\pi i G_{2k+2}(z)(X-zY)^{2k}\,dz) = \pi i G_{2k+2}(z)(X-zY)^{2k}\,dz - \pi i\overline{G_{2k+2}(z)}(X-\overline{z}Y)^{2k}\,d\overline{z}. \tag{7}$$

In fact, $\mathcal{E}_{2k}$ is uniquely determined by (6) and (7) up to an additive constant, but we will not prove this here.

Proof. The differential equation is clear from the definition of $\mathcal{E}_{2k}(z)$. For modular invariance, it is enough to very this for $\gamma \in \{S, T\}$. To check this, we make use of Proposition 6.5 and get

$$\begin{aligned}
\mathcal{E}_{2k}(T.z)|T &= (2\pi i)^{-2k}\,\mathrm{Re}\left(I_{G_{2k+2}}(T.z)|T + \frac{(2k)!}{2}\zeta(2k+1)Y^{2k}|T\right)\\
&= (2\pi i)^{-2k}\,\mathrm{Re}\left(I_{G_{2k+2}}(z) + \frac{(2k)!}{2}\zeta(2k+1)Y^{2k}\right)\\
&= \mathcal{E}_{2k}(z)
\end{aligned}$$

since the monodromy under T is purely imaginary and since T acts trivially on Y. Likewise, we get using Proposition 6.5

$$\begin{aligned}
\mathcal{E}_{2k}(S.z)|S &= (2\pi i)^{-2k}\,\mathrm{Re}\left(I_{G_{2k+2}}(S.z)|S + \frac{(2k)!}{2}\zeta(2k+1)Y^{2k}|S\right)\\
&= (2\pi i)^{-2k}\,\mathrm{Re}\left(I_{G_{2k+2}}(z) + \frac{(2k)!}{2}\zeta(2k+1)(Y^{2k} - X^{2k}) + \frac{(2k)!}{2}\zeta(2k+1)X^{2k}\right)\\
&= \mathcal{E}_{2k}(z).
\end{aligned}$$

□

We view the function $\mathcal{E}_{2k} : \mathfrak{H} \to V_{2k+2} \otimes \mathbb{C}$ as the analogue for Eisenstein series of the function $z \mapsto \log|z|^2$. In the next section we will show that $\mathcal{E}_{2k}$ is indeed the natural choice of primitive by relating it to real analytic Eisenstein series.

Remark 6.8. The cohomological explanation for Proposition 6.7 is that the map $\mathrm{SL}_2(\mathbb{Z}) \to V_k \otimes \mathbb{C}$, $\gamma \mapsto \mathrm{Re}(I_{G_{2k+2}}(\gamma.z)|\gamma - I_{G_{2k+2}}(z)) = \frac{(2k)!}{2}Y^{2k}|\gamma$ is a coboundary, i.e. its class in $H^1(\mathrm{SL}_2(\mathbb{Z}), V_{2k+2})$ vanishes. See Appendix B for more details.

6.4. **Relation to real analytic Eisenstein series.**

Definition 6.9. For $r, s \geq 0$, define the real analytic Eisenstein series of weights (r, s) and total weight $w = r + s$ by

$$\mathcal{E}_{r,s}(z) = \frac{w!}{(2\pi i)^{w+2}} \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\mathbb{L}}{(mz+n)^{r+1}(m\overline{z}+n)^{s+1}},$$

where $\mathbb{L} := \log|q| = -2\pi y$.

These functions are real analytic relatives of the holomorphic Eisenstein series $G_k(z)$. For the same reason as in the holomorphic case, we have $\mathcal{E}_{r,s} \equiv 0$ if $r+s$ is odd. Writing $w = 2k$, we have for $r = s = k$,

$$\mathcal{E}_{k,k} = \frac{i}{(2\pi i)^{2k+1}} \frac{(2k)!}{y^k} E(z, k+1),$$

where $E(z,s) = \frac{1}{2}\sum'_{(m,n)\in\mathbb{Z}^2} \frac{y^s}{|mz+n|^{2s}}$ is the classical real analytic Eisenstein series introduced by Hecke.

Remark 6.10. In the terminology of [4], $\mathcal{E}_{r,s}(z)$ is a non-holomorphic modular form of weights (r, s). This means that $z \mapsto \mathcal{E}_{r,s}(z)$ is a real analytic function which transforms under the action of $\mathrm{SL}_2(\mathbb{Z})$ as

$$\mathcal{E}_{r,s}(\gamma.z) = (cz+d)^r(c\overline{z}+d)^s\mathcal{E}_{r,s}(z), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Furthermore, there is also a growth condition as $y = \mathrm{Im}(z) \to \infty$ which can be phrased as a condition on the Fourier expansion of $\mathcal{E}$ (see [4], Section 2.1). In [4], the space of non-holomorphic modular forms of weights (r, s) is denoted by $\mathcal{M}_{r,s}$.

Proposition 6.11. *We have*

$$\mathcal{E}_{2k}(z) = \sum_{r+s=2k} \mathcal{E}_{r,s}(z)(X - zY)^r(X - \overline{z}Y)^s.$$

Therefore the real analytic Eisenstein series $\mathcal{E}_{r,s}(z)$ of a fixed weight $r + s = 2k$ can be assembled into a modular invariant function $z \mapsto \mathcal{E}_{2k}(z)$ which solves the differential equation (7) and which is equivalent to the following system of equations for the $\mathcal{E}_{r,s}(z)$

$$\begin{aligned} \partial\mathcal{E}_{w,0} &= \mathbb{L}G_{w+2} \\ \partial\mathcal{E}_{r,s} &= (r+1)\mathcal{E}_{r+1,s-1}, \quad \text{for all } 1 \leq s \leq w \end{aligned}$$

and

$$\begin{aligned} \overline{\partial}\mathcal{E}_{w,0} &= \mathbb{L}\overline{G}_{w+2} \\ \overline{\partial}\mathcal{E}_{r,s} &= (s+1)\mathcal{E}_{r-1,s+1}, \quad \text{for all } 1 \leq r \leq w \end{aligned}$$

Here, $\partial = \bigoplus_r \partial_r, \overline{\partial} = \bigoplus_s \overline{\partial}_s$ denote the graded differential operators on non-holomorphic modular forms, given in weights (r, s) by

$$\begin{aligned} \partial_r &= (z - \overline{z})\frac{\partial}{\partial z} + r \\ \overline{\partial}_s &= (\overline{z} - z)\frac{\partial}{\partial \overline{z}} + s. \end{aligned}$$

Remark 6.12. It is a general principle that properties of $\mathcal{E}_{2k}$ can be translated into properties of its coefficients $\mathcal{E}_{r,s}$ and vice versa. As another example, modular invariance of $\mathcal{E}_{2k}$, (6) is equivalent to $\mathcal{E}_{r,s}(z)$ being modular of weights (r, s), for all $r + s = 2k$. See [4], Proposition 7.2, for more precise statements.

Two further properties of real analytic Eisenstein series are worth mentioning. Firstly, they are eigenfunctions

$$\Delta\mathcal{E}_{r,s} = -2k\mathcal{E}_{r,s}, \quad 2k = r + s$$

for the graded hyperbolic Laplacian $\Delta = \bigoplus_{r,s} \Delta_{r,s}$, given in weight (r, s) by

$$\Delta_{r,s} = -\overline{\partial}_{s-1}\partial_r + r(s-1) = -\partial_{r-1}\overline{\partial}_s + s(r-1).$$

Secondly, the Fourier expansion of $\mathcal{E}_{r,s}$ can be computed explicitly which prominently features the odd Riemann zeta value $\zeta(2k+1)$. See [4], Section 4.1-4.2.

6.5. **Modular invariant primitives of cusp forms.** So far we have constructed modular primitives of Eisenstein series and related them to a well-known class of functions, namely the real analytic Eisenstein series. The analogous construction of modular invariant primitives of cusp forms is more elaborate and has some new features. We only indicate some parts of the construction, and refer to the original article [6] for full details.

Recall that, as explained in Remark 6.8, the key reason why the construction of modular invariant primitives of Eisenstein series worked was that the monodromy of $\mathrm{Re}(I_{G_{2k}})$ was trivial in cohomology. The analogous result is false for cusp forms, essentially by the Eichler–Shimura theorem, and this is one reason why the construction of modular invariant primitives in that case is more complicated.[3] Even stronger, for every cusp form $f \in S_k$, the differential equation

$$\frac{\partial}{\partial z}F(z) = f(z),$$

has no solution in the space $\mathcal{M}$ of non-holomorphic modular forms, [4].

On the other hand, modular invariant primitives of cusp forms can be constructed in the bigger space $\mathcal{M}^!$ of non-holomorphic modular forms which are allowed to have poles at $i\infty$. The main result is as follows.

Theorem 6.13 (Theorem 1.1. in [6])**.** *For every cuspidal Hecke eigenform of weight k for $\mathrm{SL}_2(\mathbb{Z})$, there exists a unique family*

$$\mathcal{H}(f)_{r,s} \in \mathcal{M}^!_{r,s}, \quad \textit{for all } r + s = k,\ r, s \geq 0$$

satisfying the system of differential equations

$$\partial\mathcal{H}(f)_{r,s} = (r+1)\mathcal{H}(f)_{r+1,s-1}, \quad \textit{if } s \geq 1,$$
$$\overline{\partial}\mathcal{H}(f)_{r,s} = (s+1)\mathcal{H}(f)_{r-1,s+1}, \quad \textit{if } r \geq 1,$$

and

$$\partial\mathcal{H}(f)_{k,0} = \mathbb{L}f, \quad \overline{\partial}\mathcal{H}(f)_{0,n} = \mathbb{L}\mathbf{s}(f).$$

The $\mathcal{H}(f)_{r,s}$ are eigenfunctions of the Laplace operator with eigenvalue k. Equivalently, the $\mathbb{L}^1\mathcal{H}(f)_{r,s}$ are harmonic: $\Delta\mathbb{L}^1\mathcal{H}(f)_{r,s} = 0$.

[3]This is easier to see with the version of Eichler–Shimura stated in Appendix B.3.

We do not attempt to explain the proof of the above result which occupies most of [6]. However, let us say a few words about the element $\mathbf{s}(f)$ which is one of the main differences to the case of Eisenstein series. It equals

$$\mathbf{s}(f) = \left(\frac{\eta_f^+ \omega_f^- + \eta_f^- \omega_f^+}{\eta_f^- \omega_f^+ + \eta_f^+ \omega_f^-} \right) f + \left(\frac{2\omega_f^+ \omega_f^-}{\eta_f^+ \omega_f^- \eta_f^- \omega_f^+} \right) f',$$

where $\omega_f^\pm$ are the periods of f, $\eta_f^\pm$ its quasi-periods and f' is a certain weak Hecke eigencusp form naturally associated to f.

We end by mentioning that the functions $\mathcal{H}(f)_{r,s}$ constructed in Theorem 6.13 are also closely related to weak harmonic Maass forms of integer weight, and in particular to weak harmonic lifts. The construction also gives a conceptual explanation for algebraicity of Fourier coefficients of certain mock modular forms associated to modular forms with complex multiplication. For all of this, we refer to Brown's original work [6].

Appendix A. Eisenstein series and their Fourier expansion

In this appendix we describe how to compute the Fourier expansion of several variants of Eisenstein series. We first describe the basic tool for doing this. The reference is [22].

A.1. The Poisson summation formula. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a continuous function which satisfies the growth condition $\varphi(x) = O(|x|^{-c})$ for some $c > 1$ if $|x| \to \infty$. Then the function $\Phi : \mathbb{R} \to \mathbb{C}$ given by

$$\Phi(x) = \sum_{n \in \mathbb{Z}} \varphi(x+n)$$

is well-defined and continuous (since the above sum converges absolutely and locally uniformly). Moreover, it is clearly one-periodic, $\Phi(x+1) = \Phi(x)$ and therefore has a Fourier expansion $\Phi(x) = \sum_{r \in \mathbb{Z}} a_r e^{2\pi i r x}$, with coefficients $a_r = \int_0^1 \Phi(t) e^{-2\pi i r t} \mathrm{d}t$.

Proposition A.1 (Poisson summation)**.** *We have*

$$\Phi(x) = \sum_{r \in \mathbb{Z}} \left(\int_{\mathbb{R}} \varphi(t) e^{-2\pi i r t} dt \right) e^{2\pi i r x}.$$

Proof. By what was said above, the formula is equivalent to

$$a_r = \int_{\mathbb{R}} \varphi(t) e^{-2\pi i r t} \mathrm{d}t.$$

Indeed, we have

$$\begin{aligned} a_r = \int_0^1 \Phi(t) e^{-2\pi i r t} \mathrm{d}t = \int_0^1 \sum_{n \in \mathbb{Z}} \varphi(t+n) e^{-2\pi i r (t+n)} \mathrm{d}t &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} \varphi(t) e^{-2\pi i r t} \mathrm{d}t \\ &= \int_{\mathbb{R}} \varphi(t) e^{-2\pi i r t} \mathrm{d}t. \end{aligned}$$

□

In the remainder of this section we give two applications of this result.

A.2. **Holomorphic Eisenstein series.** As a first application of Poisson summation, we will compute the Fourier expansion of the non-normalized Eisenstein series

$$\mathbb{G}_{2k}(z) = \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(mz+n)^{2k}}.$$

which is related to the Hecke-normalized Eisenstein series introduced in Section 1 by $\mathbb{G}_{2k}(z) = \frac{2(2\pi i)^{2k-1}}{(2k-1)!} G_{2k}(z)$. Splitting the sum into terms with $m = 0$ and $m \neq 0$ respectively, we get

$$\begin{aligned} \mathbb{G}_{2k}(z) &= \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^{2k}} + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^{2k}} \\ &= 2\zeta(2k) + 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^{2k}}. \end{aligned} \tag{8}$$

To proceed further, we now establish the following result known as Lipschitz' formula.

Proposition A.2. *For every $k \geq 2$ and $z = x+iy$ with $y > 0$, we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty} r^{k-1} q^r.$$

Proof. Applying the Poisson summation formula to the function $x \mapsto (x+iy)^{-2k}$ ($y > 0$ fixed) gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{(x+iy+n)^{2k}} = \sum_{r\in\mathbb{Z}} \left(\int_{\mathbb{R}} \frac{e^{-2\pi i r t}}{(t+iy)^{2k}} \mathrm{d}t\right) e^{2\pi i r x} = \sum_{r\in\mathbb{Z}} \left(\int_{-\infty+iy}^{\infty+iy} \frac{e^{-2\pi i r t}}{t^{2k}} \mathrm{d}t\right) e^{2\pi i r z}.$$

The integrals in the final sum can now be evaluated using the residue theorem and this gives

$$\int_{-\infty+iy}^{\infty+iy} \frac{e^{-2\pi i r t}}{t^{2k}} \mathrm{d}t = \begin{cases} 0 & r \leq 0 \\ \frac{(-2\pi i)^{2k}}{(2k-1)!} r^{2k-1} & r > 0, \end{cases}$$

from which the result follows. ☐

Plugging Proposition A.2 into (8), we get

$$\mathbb{G}_{2k}(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where $\sigma_l(n) = \sum_{d|n} d^l$. In particular, we also obtain

$$\begin{aligned} G_{2k}(z) = \frac{(2k-1)!}{2(2\pi i)^{2k}} \mathbb{G}_{2k}(z) &= \frac{\zeta(2k)(2k-1)!}{(2\pi i)^{2k}} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \\ &= -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \end{aligned}$$

where in the last equation we used Euler's theorem

$$\zeta(2k) = -\frac{B_{2k}(2\pi i)^{2k}}{2(2k)!}, \quad k \geq 1,$$

which can also be proved using Poisson summation (see for example [14], Section III.3).

A.3. **Real analytic Eisenstein series.** As a final example, consider the real analytic Eisenstein series

$$E(z,s) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{y^s}{|mz+n|^{2s}},$$

defined for $z \in \mathfrak{H}$ and $\mathrm{Re}(s) > 1$. It is related to the real analytic Eisenstein series $\mathcal{G}_0^s(z)$ introduced in Section 6.4 via the formula $E(z,s) = (\zeta(2s))^{-1}\mathcal{G}_0^s(z)$. Computing its Fourier expansion can again be done using the Possion summation formula but the execution is somewhat more involved than for $\mathbb{G}_{2k}(z)$. We refer to [22], Section 3A for the details and just state the final result:

$$E(z,s) = \zeta(2s)y^s + \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)}y^{1-s} + \frac{2\pi^s}{\Gamma(s)}y^{\frac{1}{2}} \sum_{n\neq 0} \sigma^*_{s-\frac{1}{2}}(|n|)K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx},$$

where $K_\nu(t) = \int_0^\infty e^{-t\cosh(u)}\cosh(\nu u)du$ for $\nu \in \mathbb{C}$ and $t > 0$ denotes the K-Bessel function and $\sigma^*_\nu(n) := |n|^\nu \sum_{d|n} d^{-2\nu}$ is a modified divisor function.

An important consequence is that the function $s \mapsto G(z,s)$ can be meromorphically continued to the entire complex plane and the completed Eisenstein series $G^*(z,s) := \pi^{-s}\Gamma(s)G(z,s)$ satisfies a functional equation. We summarize this and the Fourier expansion in the following proposition.

Proposition A.3. *The function $G^*(z,s)$ has analytic continuation to $\mathbb{C}\setminus\{0,1\}$ and has simple poles at $s = 1$, $s = 0$ with residues $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Furthermore, it satisfies the functional equation $G^*(z,s) = G^*(z,1-s)$ and has the Fourier expansion*

$$G^*(z,s) = \zeta^*(2s)y^s + \zeta^*(2s-1)y^{1-s} + 2y^{\frac{1}{2}} \sum_{n\neq 0} \sigma^*_{s-\frac{1}{2}}(|n|)K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx},$$

where $\zeta^*(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

Proof. Both the meromorphic continuation as well as the location and residues of the poles follow from the fact that $\sigma^*_\nu(t)$ and $K_\nu(t)$ are entire functions of ν and that $\zeta^*(s)$ has a meromorphic continuation to the complex plane with simple poles at $s = 0$, $s = 1$ with residues -1 and 1 respectively. The functional equation of $G^*(z,s)$ likewise follows from the functional equation $\zeta^*(1-s) = \zeta^*(s)$ together with $\sigma^*_{-\nu}(t) = \sigma^*_\nu(t)$ and $K_{-\nu}(t) = K_\nu(t)$, both of which are immediate from the definition. □

Remark A.4. The Fourier expansion of G^* can be used to prove Kronecker's first limit formula

$$\lim_{s\to 1}\left(G^*(z,s) - \frac{1/2}{s-1}\right) = -\frac{1}{24}\log(y^{12}|\Delta(z)|^2) + C,$$

where $C = \frac{1}{2}\gamma - \frac{1}{2}\log 4\pi$, ($\gamma$ =Euler's constant). Together with modular invariance of $G^*(z,s)$, it gives a new proof that $\Delta(z)$ is a modular form of weight 12. However, its main applications are in number theory where it can be used to construct solutions to Pell's equation. For this and more, we refer to [16].

Appendix B. Cohomological approach to periods of modular forms

In this appendix we interpret some of the structures we encountered so far in our study of periods of modular forms using the language of group cohomology. In particular, we rephrase the Eichler–Shimura theorem as a statement which relates modular forms to group cohomology of $\mathrm{SL}_2(\mathbb{Z})$ with coefficients in spaces of homogeneous polynomials.

To this end, we begin by presenting the bare minimum of the language of group cohomology required for our purposes, in particular several important concepts, such as relative cohomology or cup products, have been omitted. We refer to [18] for an extensive and very systematic introduction and to [1] for a quick, if perhaps slightly outdated, one.

B.1. **Generalities on group cohomology.** Let G be a group and V be a right G-module, i.e. V is a $\mathbb{Z}$-module (aka abelian group) together with a right action

$$V \times G \to V, \quad (v, g) \mapsto v|g,$$

of G which is compatible with the $\mathbb{Z}$-module structure of V. To these, one can associate cohomology groups $\{H^i(G;V)\}_{i\geq 0}$, [18]. To compute these groups, consider the following cochain complex

$$\ldots \longrightarrow C^{i-1}(G;V) \xrightarrow{d^{i-1}} C^i(G;V) \xrightarrow{d^i} C^{i+1}(G;V) \longrightarrow \ldots.$$

Here $C^i(G;V) = \{\varphi : G^i \to V\}$ is the set of all maps from the i-fold cartesian product of G into V, and the differential in degree i is given by

$$(d^i\varphi)(g_1,\ldots,g_{i+1}) = \varphi(g_1,\ldots,g_i)|g_{i+1} + \sum_{j=0}^{i-1}(-1)^{j+1}\varphi(g_1,\ldots,g_{i-j}g_{i-j+1},\ldots,g_{i+1}) + (-1)^{i+1}\varphi(g_1,\ldots,g_i).$$

Elements of $\ker(d)$ (respectively of $\mathrm{Im}(d)$) are called cocycles (respectively coboundaries), and one has

$$H^i(G;V) \cong \ker(d^i)/\operatorname{Im}(d^{i-1}).$$

In the cases $i = 0, 1$, which will be the only cases of interest for us, we have the following very explicit descriptions. The group $H^0(G;V)$ is simply the group of invariants

$$H^0(G;V) \cong V^G := \{v \in V \mid v|g = v,\ \forall g \in G\},$$

and $H^1(G;V)$ is isomorphic to the quotient of the group of *crossed homomorphisms*

$$\ker(d^1) = \{f : G \to V \mid f(g_1g_2) = f(g_1)|g_2 + f(g_2),\ \forall g_1, g_2 \in G\}$$

by the subgroup of those coming from elements of V,

$$\operatorname{Im}(d^0) = \{f : G \to V \mid \exists v \in V,\ f(g) = v|g,\ \forall g \in G\}.$$

B.2. **The case** $\mathrm{SL}_2(\mathbb{Z})$**.** For us the case of interest will be $G = \mathrm{SL}_2(\mathbb{Z})$ and the coefficients will be $V_k = \bigoplus_{0\leq n\leq k-2} \mathbb{Q}X^nY^{k-2-n}$ for some even integer $k \geq 2$ with its $\mathrm{SL}_2(\mathbb{Z})$-right action as described in Section 3.1. In this case, we have

$$H^0(\mathrm{SL}_2(\mathbb{Z}); V_k) \cong V_k^{\mathrm{SL}_2(\mathbb{Z})} \begin{cases} \mathbb{Q} & k = 2 \\ \{0\} & k > 2 \end{cases}, \quad \text{and } H^i(\mathrm{SL}_2(\mathbb{Z}); V_k) \cong \{0\},\ i \geq 2.$$

For the first statement, note that a polynomial $f \in V_k$ is invariant under the action of the translation T if and only if $f = \alpha X^{k-2}$ for some $\alpha \in \mathbb{Q}$, and that $\alpha X^{k-2}|S = \alpha(-Y)^{k-2}$, so that $\alpha = 0$ if $f \in V_k^{\mathrm{SL}_2(\mathbb{Z})}$, for $k > 2$. The second statement can be shown using geometric reasoning, namely because $\mathrm{SL}_2(\mathbb{Z})$ can be realized as the fundamental group of a non-compact Riemann surface. Therefore, the only group of interest is $H^1(\mathrm{SL}_2(\mathbb{Z}), V_k)$ which will be described in the next two subsections using modular forms.

B.3. **The Eichler–Shimura isomorphism revisited.** Recall from Section 6.2 that to a modular form $f \in M_k$, we have associated the indefinite integral $I_f(z)$ which takes values in $V_k \otimes \mathbb{C}$.

Proposition B.1. *The map*

$$\varphi_f : \mathrm{SL}_2(\mathbb{Z}) \to V_k \otimes \mathbb{C}, \quad \gamma \mapsto I_f(\gamma.z)|\gamma - I_f(z).$$

does not depend on the choice of z and is a right $\mathrm{SL}_2(\mathbb{Z})$*-cocycle.*

In fact, this map already appeared in Section 6.2 as the monodromy of the integral $I_f(z)$.

Proof. It follows from $\mathrm{SL}_2(\mathbb{Z})$-invariance of $\underline{f}$ that both $I_f(z)$ and $I_f(\gamma.z)|\gamma$ are solutions to the differential equation $\mathrm{d}g = -\underline{f}$, and therefore differ by a constant. For the cocycle equation, by definition we have

$$\varphi_f(\gamma_1\gamma_2) = I_f((\gamma_1\gamma_2).z)|_{\gamma_1\gamma_2} - I_f(z).$$

Since $I_f(z) = -\varphi_f(\gamma_2) + I_f(\gamma_2.z)|_{\gamma_2}$, the preceding equality becomes

$$\begin{aligned}\varphi_f(\gamma_1\gamma_2) &= I_f((\gamma_1\gamma_2).z)|_{\gamma_1\gamma_2} - I_f(\gamma_2.z)|_{\gamma_2} + \varphi_f(\gamma_2) \\ &= \varphi_f(\gamma_1)|_{\gamma_2} + \varphi_f(\gamma_2).\end{aligned}$$

□

Since the assignment $f \mapsto I_f(z)$ is clearly $\mathbb{C}$-linear, the preceding proposition implies that we get a map

$$[\psi] : M_k \to H^1(\mathrm{SL}_2(\mathbb{Z}), V_k) \otimes \mathbb{C}, \quad f \mapsto [\varphi_f],$$

of $\mathbb{C}$-vector spaces, where $[\varphi_f]$ denotes the cohomology class of the cocycle φ_f.

Now conjugation with the element $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on V_k induces an involution on the level of cochain complexes

$$C^\bullet(\mathrm{SL}_2(\mathbb{Z}), V_k) \to C^\bullet(\mathrm{SL}_2(\mathbb{Z}), V_k)$$

and therefore we can split the cohomology into the corresponding ±1-eigenspaces under that involution,

$$H^1(\mathrm{SL}_2(\mathbb{Z}), V_k) = H^1(\mathrm{SL}_2(\mathbb{Z}), V_k)^+ \oplus H^1(\mathrm{SL}_2(\mathbb{Z}), V_k)^-.$$

Also, let $M_k^{\mathbb{R}} \subset M_k$ be the $\mathbb{R}$-vector subspace of modular forms with real Fourier coefficients, and define $S_k^{\mathbb{R}} \subset S_k$ likewise. With this, we can finally state the cohomological version of Theorem 4.2.

Theorem B.2 (Eichler–Shimura–Manin)**.** *The morphisms*

$$[\mathrm{Re}(\psi)] : S_k^{\mathbb{R}} \to H^1(\mathrm{SL}_2(\mathbb{Z}), V_k)^+ \otimes \mathbb{R}$$
$$[\mathrm{Im}(\psi)] : M_k^{\mathbb{R}} \to H^1(\mathrm{SL}_2(\mathbb{Z}), V_k)^- \otimes \mathbb{R},$$

are isomorphisms of $\mathbb{R}$*-vector spaces.*

Remark B.3. (i) That $[\mathrm{Re}(\psi)]$ is well-defined follows from from the even periods of $f \in S_k^{\mathbb{R}}$ being real numbers. Indeed,

$$(2\pi i)^{k-1} p_n(f) = (2\pi i)^{k-1} \int_0^{i\infty} f(z) z^n \mathrm{d}z \in \mathbb{R},$$

since f has real Fourier coefficients. A similar argument combined with Proposition 6.5 (for the case of Eisenstein series) shows that $[\mathrm{Im}(\psi)]$ is well-defined.

(ii) Theorem B.2 says in particular that $[\mathrm{Re}(\psi_{G_{2k}})] = 0$, i.e. $\mathrm{Re}(\psi_f)$ is a coboundary if f is an Eisenstein series, which also follows from Proposition 6.5. This is the key reason why the construction of the modular invariant primitive of G_{2k} in Section 6.3 worked.

References

[1] M.F. Atiyah, C.T.C. Wall, *Cohomology of groups.* Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 94–115, Thompson, Washington, D.C., 1967.

[2] B.C. Berndt, A. Straub, *Ramanujan's formula for* $\zeta(2n+1)$. Exploring the Riemann zeta function, 1334, Springer, Cham, 2017.

[3] F. Brown, *Multiple modular values and the relative completion of the fundamental group of* $\mathcal{M}_{1,1}$. arXiv:1407.5167.

[4] F. Brown, *A class of non-holomorphic modular forms I.* Res. Math. Sci. 5 (2018), no. 1, Paper No. 7, 40 pp.

[5] F. Brown, *A class of non-holomorphic modular forms II: Equivariant iterated Eisenstein integrals.* arXiv:1708.03354.

[6] F. Brown, *A class of non-holomorphic modular forms III: real analytic cusp forms for* $\mathrm{SL}_2(\mathbb{Z})$. Res. Math. Sci. 5 (2018), no. 3, Paper No. 34, 36 pp.

[7] D. Bump, *Automorphic forms and representations.* Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997. xiv+574 pp.

[8] F. Diamond, J. Shurman, *A first course in modular forms.* Graduate Texts in Mathematics, 228. Springer-Verlag, New York, 2005. xvi+436 pp.

[9] K. Haberland, *Perioden von Modulformen einer Variabler and Gruppencohomologie, I.* Math. Nachr. 112 (1983), 245–282.

[10] M. Hirose, N. Sato, K. Tasaka, *Eisenstein series identities based on partial fraction decomposition.* Ramanujan J. 38 (2015), no. 3, 455–463.

[11] W. Kohnen, D. Zagier, *Modular forms with rational periods.* Modular forms (Durham, 1983), 197–249, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, 1984.

[12] M. Kontsevich, D. Zagier, *Periods.* Mathematics unlimited–2001 and beyond, 771–808, Springer, Berlin, 2001.

[13] S. Lang, *Introduction to modular forms.* Grundlehren der mathematischen Wissenschaften, No. 222. Springer-Verlag, Berlin–New York, 1976. ix+261 pp.

[14] B. Schoeneberg, *Elliptic modular functions: an introduction.* Translated from the German by J. R. Smart and E. A. Schwandt. Die Grundlehren der mathematischen Wissenschaften, Band 203. Springer-Verlag, New York-Heidelberg, 1974. viii+233 pp.

[15] J.-P. Serre, *Cours d'arithmétique.* Collection SUP: "Le Mathématicien", 2 Presses Universitaires de France, Paris 1970, 188 pp.

[16] C.-L. Siegel, *Lectures on advanced analytic number theory.* Notes by S. Raghavan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 23 Tata Institute of Fundamental Research, Bombay 1965 iii+331+iii pp.

[17] J. Sturm, *On the congruence of modular forms.* Number theory (New York, 1984–1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin, 1987.

[18] C. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp.

[19] D. Zagier, *The Rankin–Selberg method for automorphic functions which are not of rapid decay.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 415–437 (1982).

[20] D. Zagier, *Periods of modular forms and Jacobi theta functions.* Invent. Math. **104** (1991), no. 3, 449–465.

[21] D. Zagier, *Modular forms of one variables (Notes based on a course given Utrecht, Spring 1991).* Available under: `https://people.mpim-bonn.mpg.de/zagier/files/tex/UtrechtLectures/UtBook.pdf`

[22] D. Zagier, *Introduction to modular forms.* From number theory to physics (Les Houches, 1989), 238–291, Springer, Berlin, 1992.

[23] D. Zagier, *Elliptic modular forms and their applications.* The 1-2-3 of modular forms, 1–103, Universitext, Springer, Berlin, 2008.

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom

Email address: `nils.matthes@maths.ox.ac.uk`