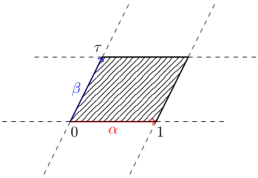
Elliptic multiple zeta values



$$\int_{\alpha} \bigwedge^{n} \omega_{\mathrm{KZB}} = \sum_{k_1,\ldots,k_n \geq 0} I^{\mathrm{A}}(k_1,\ldots,k_n;\tau) X_1^{k_1-1} \ldots X_n^{k_n-1}$$

$$(2\pi i)^2 I^{\mathrm{A}}(0,1,0,0;\tau) = -3\zeta(3) + 6q + \frac{27}{4}q^2 + \frac{56}{9}q^3 + \ldots$$

Nils Matthes

Fachbereich Mathematik
Universität Hamburg

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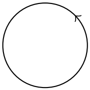
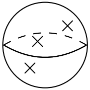
Introduction

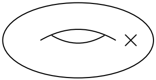
Common theme in number theory/arithmetic geometry: associate intrinsic numbers ("periods") to geometric objects



S^1 $\xleftrightarrow{\int \frac{\mathrm{d}z}{z}}$ $2\pi i$



$\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ $\longleftrightarrow$ multiple zeta values



$\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \setminus \{0\}$ $\longleftrightarrow$ elliptic multiple zeta values

Multiple zeta values

Definition

For natural numbers $k_1, \ldots, k_{n-1},\ k_n \geq 2$ define the multiple zeta value

$$\zeta(k_1, \ldots, k_n) = \sum_{0<m_1<\ldots<m_n} \frac{1}{m_1^{k_1} \ldots m_n^{k_n}} \in \mathbb{R}.$$

Also, let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space spanned by all multiple zeta values.

Proposition

$\mathcal{Z}$ is a $\mathbb{Q}$-subalgebra of $\mathbb{R}$.

Deligne, Drinfeld, Kontsevich ...

Multiple zeta values arise naturally from the monodromy of the Knizhnik-Zamolodchikov connection ∇_{KZ} on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$.

Geometry of multiple zeta values I

Let x_0, x_1 formal, non-commuting variables, $\mathbb{C}\langle\langle x_0, x_1\rangle\rangle$ formal power series.

Knizhnik-Zamolodchikov connection

$$\nabla_{\mathrm{KZ}} := \mathrm{d} - \omega_{\mathrm{KZ}}, \quad \omega_{\mathrm{KZ}} = \omega_0 \cdot x_0 + \omega_1 \cdot x_1,$$

where $\omega_0 = \frac{\mathrm{d}z}{z}$, $\omega_1 = \frac{\mathrm{d}z}{z-1}$.

This connection is integrable: $\nabla_{\mathrm{KZ}}^2 = 0$.

Monodromy map:

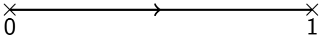
$$T_{a,b}^{\mathrm{KZ}} : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}; b, a) \to \mathbb{C}\langle\langle x_0, x_1\rangle\rangle,$$

$$\gamma \mapsto 1 + \sum_{n=1}^{\infty} \int_\gamma \bigwedge^n \omega_{\mathrm{KZ}},$$

for all $a, b \in \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$.

Geometry of multiple zeta values II

The canonical path from 0 to 1 is called “droit chemin” (dch).



The monodromy of ∇_{KZ} along dch diverges, but can be regularized.

Definition (Drinfeld associator)

$$\Phi_{\mathrm{KZ}}(x_0, x_1) := (T_{0,1}^{\mathrm{KZ}})^{\mathrm{reg}}(\mathrm{dch}) \in \mathbb{C}\langle\langle x_0, x_1 \rangle\rangle.$$

Proposition (Kontsevich, Le–Murakami)

We have

$$\mathcal{Z} = \mathrm{Span}_{\mathbb{Q}}\{\Phi_{\mathrm{KZ}}(x_0, x_1)\big|_w \mid w \in \langle x_0, x_1 \rangle\}.$$

More precisely, for $w = x_0^{k_n-1}x_1 \dots x_0^{k_1-1}x_1$, and $k := k_1 + \dots + k_n$, we have

$$\Phi_{\mathrm{KZ}}(x_0, x_1)\big|_w = \int\limits_{1 \geq t_1 \geq \dots \geq t_k \geq 0} \left(\bigwedge^{k_n-1} \omega_0\right) \wedge \omega_1 \wedge \dots \wedge \left(\bigwedge^{k_1-1} \omega_0\right) \wedge \omega_1$$
$$= (-1)^n \zeta(k_1, \dots, k_n).$$

The elliptic KZB connection (Brown–Levin, Calaque–Enriquez–Etingof, Levin–Racinet,. . .)

Let

$$E_\tau^\times := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \setminus \{0\}$$

be the once-punctured complex elliptic curve with modulus $\tau \in \mathfrak{H}$ ($\mathfrak{H}$ the upper half-plane).

Definition (Brown–Levin)

The *elliptic Knizhnik-Zamolodchikov-Bernard (KZB) connection* is defined as

$$\nabla_{\mathrm{KZB}} := \mathrm{d} - \omega_{\mathrm{KZB}}, \quad \omega_{\mathrm{KZB}} = 2\pi i \mathrm{d}r \cdot a - \sum_{k=0}^{\infty} \omega^{(k)} \operatorname{ad}^k(a)(b) \in \Omega^1(E_\tau^\times)\widehat{\otimes}\mathbb{C}\langle\langle a, b\rangle\rangle,$$

Differential forms

The differential forms $\omega^{(k)} \in \Omega^1(E_\tau^\times)$ are defined as coefficients of the Laurent expansion

$$e^{2\pi i r\alpha}\frac{\theta_\tau'(0)\theta_\tau(\xi+\alpha)}{\theta_\tau(\xi)\theta_\tau(\alpha)}\mathrm{d}\xi = \sum_{k=0}^{\infty}\omega^{(k)}\alpha^{k-1},$$

at $\alpha = 0$, where $\xi = s + r\tau$ is the coordinate on $E_\tau^\times$.

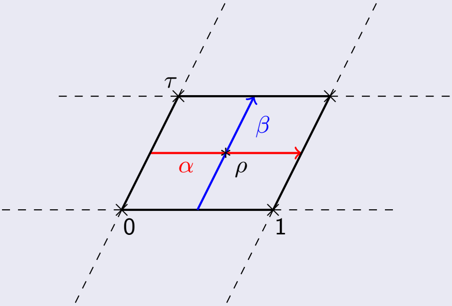
Monodromy of ∇_{KZB}

Since $\nabla^2_{\mathrm{KZB}} = 0$, we get a monodromy representation

$$T^{\mathrm{KZB}}_{\rho} : \pi_1(E^\times_\tau; \rho) \to \mathbb{C}\langle\langle a, b\rangle\rangle$$
$$\gamma \mapsto 1 + \sum_{n=1}^{\infty} \int_\gamma \bigwedge^n \omega_{\mathrm{KZB}},$$

for every $\rho \in E^\times_\tau$. It is uniquely determined on the generators α, β of $\pi_1(E^\times_\tau; \rho)$.

A picture of $E^\times_\tau$



The elliptic KZB associator

For $\rho \to 0$, $T_\rho^{\mathrm{KZB}}(\alpha)$, $T_\rho^{\mathrm{KZB}}(\beta)$ diverge, but can be regularized.

Definition (Enriquez)

Define $A, B : \mathfrak{H} \to \mathbb{C}\langle\langle a, b\rangle\rangle$ to be the regularized limits

$$A(\tau) = \lim_{\rho\to 0} T_\rho^{\mathrm{KZB}}(\alpha), \quad B(\tau) = \lim_{\rho\to 0} T_\rho^{\mathrm{KZB}}(\beta).$$

Enriquez's *elliptic KZB associator* is the triple $(\Phi_{\mathrm{KZ}}, A(\tau), B(\tau))$.

Consider the $\mathbb{Q}$-vector space $\mathcal{E}\mathcal{Z}^{\mathrm{A}}$ (resp. $\mathcal{E}\mathcal{Z}^{\mathrm{B}}$) spanned by the coefficients of $A(\tau)$ (resp. of $B(\tau)$).

Proposition

Both $\mathcal{E}\mathcal{Z}^{\mathrm{A}}$ and $\mathcal{E}\mathcal{Z}^{\mathrm{B}}$ are $\mathbb{Q}$-subalgebras of $\mathcal{O}(\mathfrak{H})$.

We call $\mathcal{E}\mathcal{Z}^{\mathrm{A}}$ (resp. $\mathcal{E}\mathcal{Z}^{\mathrm{B}}$) the *algebra of A-elliptic multiple zeta values* (resp. *algebra of B-elliptic multiple zeta values*).

Algebras of elliptic multiple zeta values

Question

What can we say about the structure of $\mathcal{EZ}^{\mathrm{A}}$ and $\mathcal{EZ}^{\mathrm{B}}$?

Definition

Let $\mathfrak{A} = \{\mathbf{a}_i\}_{i\in I}$ be a set (an alphabet of letters), and define the *shuffle* $\mathbb{Q}$*-algebra* $(\mathbb{Q}\langle\mathfrak{A}\rangle, ш)$ as follows: Its underlying vector space is

$$\mathbb{Q}\langle\mathfrak{A}\rangle := \mathrm{Span}_{\mathbb{Q}}\{\mathbf{a}_{i_1}\dots\mathbf{a}_{i_n} \mid a_{i_j} \in \mathfrak{A}\}.$$

The (commutative) product is given by the shuffle product

$$ш : \mathbb{Q}\langle\mathfrak{A}\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle\mathfrak{A}\rangle \to \mathbb{Q}\langle\mathfrak{A}\rangle$$
$$\mathbf{a}_{i_1}\dots\mathbf{a}_{i_r} \otimes \mathbf{a}_{i_{r+1}}\dots\mathbf{a}_{i_{r+s}} \mapsto \sum_{\sigma\in\Sigma_{r,s}} \mathbf{a}_{i_{\sigma(1)}}\dots\mathbf{a}_{i_{\sigma(r+s)}},$$

where $\Sigma_{r,s}$ is the set of permutations of $\{1,\dots,r+s\}$, such that $\sigma^{-1}(1) < \dots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \dots < \sigma^{-1}(r+s)$.

Example: $a_i \,ш\, a_j a_k = a_i a_j a_k + a_j a_i a_k + a_j a_k a_i$.

The canonical embeddings

Now let $\mathfrak{A} = \mathfrak{E} := \{\mathbf{e}_{2k}\}_{k\geq 0}$.

Theorem (M., Broedel–M.–Schlotterer)

There exist canonical embeddings of $\mathbb{Q}$-algebras

$$\psi^A : \mathcal{E}\mathcal{Z}^{\mathrm{A}} \hookrightarrow \mathbb{Q}\langle \mathfrak{E}\rangle \otimes_{\mathbb{Q}} \mathcal{Z}[(2\pi i)^{-1}],$$
$$\psi^B : \mathcal{E}\mathcal{Z}^{\mathrm{B}} \hookrightarrow \mathbb{Q}\langle \mathfrak{E}\rangle \otimes_{\mathbb{Q}} \mathcal{Z}[(2\pi i)^{-1}].$$

Two crucial steps in the proof:

- The differential equation for $A(\tau)$ and $B(\tau)$ (due to Enriquez), which expresses their coefficients as $\mathcal{Z}[(2\pi i)^{-1}]$-linear combinations of iterated Eisenstein integrals

$$\mathcal{E}(2k_1,\dots,2k_n;\tau) := \frac{1}{(2\pi i)^n}\int_{i\infty}^{\tau} E_{2k_1}(\tau_1)\mathrm{d}\tau_1 \wedge \ldots \wedge E_{2k_n}(\tau_n)\mathrm{d}\tau_n,$$

  where E_{2k} is the holomorphic Eisenstein series of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$.
- The $\mathbb{C}$-linear independence of these iterated Eisenstein integrals (M.)

A Lie algebra of derivations

Question

Are ψ^A, ψ^B surjective?

Let $\mathcal{L} = \mathbb{L}(a, b)$ be the free Lie algebra on a, b, $\mathrm{Der}^0(\mathcal{L})$ the set of all derivations $D : \mathcal{L} \to \mathcal{L}$, such that $D([a, b]) = 0$ and $a^\vee(D(b)) = 0$.

Definition (Tsunogai, Nakamura, . . ., Hain–Matsumoto, Calaque–Enriquez–Etingof, Pollack)

(i) For every $k \geq 0$, define a derivation $\varepsilon_{2k} \in \mathrm{Der}^0(\mathcal{L})$ such that

$$\varepsilon_{2k}(a) = \mathrm{ad}^{2k}(a)(b).$$

(ii) Let $\mathfrak{u} = \mathrm{Lie}_{\mathbb{Q}}(\varepsilon_{2k}) \subset \mathrm{Der}^0(\mathcal{L})$ be the Lie subalgebra generated by the ε_{2k}.

Fact (Ihara–Takao, Pollack)

$\mathfrak{u}$ is not freely generated by the ε_{2k}.

The image of the decomposition map

The universal enveloping algebra $U(\mathfrak{u})$ is graded by giving every ε_{2k} degree one. By abstract algebra, we have an embedding

$$U(\mathfrak{u})^{\vee} \hookrightarrow \mathbb{Q}\langle\mathfrak{E}\rangle.$$

Since $\mathfrak{u}$ is not freely generated, this image is a proper subspace of $\mathbb{Q}\langle\mathfrak{E}\rangle$.

Theorem (M., Broedel–M.–Schlotterer)

The decomposition morphisms ψ^A, ψ^B factor through $U(\mathfrak{u})^{\vee}$:

$$\psi^A : \mathcal{EZ}^{\mathrm{A}} \hookrightarrow U(\mathfrak{u})^{\vee} \otimes_{\mathbb{Q}} \mathcal{Z}[(2\pi i)^{-1}],$$
$$\psi^B : \mathcal{EZ}^{\mathrm{B}} \hookrightarrow U(\mathfrak{u})^{\vee} \otimes_{\mathbb{Q}} \mathcal{Z}[(2\pi i)^{-1}].$$

Not every linear combination of iterated Eisenstein integrals is contained in $\mathcal{EZ}^{\mathrm{A}}, \mathcal{EZ}^{\mathrm{B}}$! There are non-trivial constraints coming from relations in $\mathfrak{u}$.

Relations in $\mathfrak{u}$

Example (Broedel–M.–Schlotterer)

The Ihara–Takao relation in $\mathfrak{u}$ is

$$[\varepsilon_{10}, \varepsilon_4] - 3[\varepsilon_8, \varepsilon_6] = 0.$$

Together with the preceding theorem, this implies that the double Eisenstein integrals $\mathcal{E}(8,6;\tau)$ and $\mathcal{E}(10,4;\tau)$ can occur in $\mathcal{E}\mathcal{Z}^{\mathrm{A}}$, $\mathcal{E}\mathcal{Z}^{\mathrm{B}}$ only in the combination

$$\underbrace{81}_{3\cdot(9\cdot 3)}\mathcal{E}(10,4;\tau) + \underbrace{35}_{7\cdot 5}\mathcal{E}(8,6;\tau).$$

Remark

Non-trivial relations in $\mathfrak{u}$ come from period polynomials of modular forms, for $\mathrm{SL}_2(\mathbb{Z})$ (Hain–Matsumoto, Pollack).
For example, the above Ihara–Takao relation comes from the even period polynomial of some modular form of weight 12.

A-elliptic multiple zeta values

The algebra $\mathcal{EZ}^{\mathrm{A}}$ has canonical generators.

Definition (Enriquez)

For $k_1, \ldots, k_n \geq 0$, with $k_1, k_n \neq 1$, define

$$I^{\mathrm{A}}(k_1, \ldots, k_n; \tau) = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} \alpha^*(\omega^{(k_1)})(t_1) \wedge \ldots \wedge \alpha^*(\omega^{(k_n)})(t_n).$$

This has *weight* $k_1 + \ldots + k_n$ and *length* n. We call $I^{\mathrm{A}}(k_1, \ldots, k_n; \tau)$ an *A-elliptic multiple zeta value*.

The definition can be extended to all $k_i \geq 0$.

Proposition (M.)

We have $\mathcal{EZ}^{\mathrm{A}} = \mathrm{Span}_{\mathbb{Q}}\{I^{\mathrm{A}}(k_1, \ldots, k_n; \tau)\}$.

An example

$$
\begin{aligned}
I^{\mathrm{A}}(0,1,0,0;\tau) &= -3\frac{\zeta(3)}{(2\pi i)^2} \\
&\quad + \frac{18}{(2\pi i)^3}\int_{i\infty}^{\tau} E_0(\tau_1)\mathrm{d}\tau_1 \wedge E_0(\tau_2)\mathrm{d}\tau_2 \wedge E_4(\tau_3) - 2\zeta(4)\mathrm{d}\tau_3 \\
&= -3\frac{\zeta(3)}{(2\pi i)^2} + \frac{6}{(2\pi i)^2}\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^n,
\end{aligned}
$$

where $E_0(\tau) := -1$, and $\sigma_l(n) = \sum_{d|n} d^l$.

Relations between elliptic multiple zeta values

Many algebraic relations between $I^{\mathrm{A}}(k_1, \ldots, k_n; \tau)$. E.g.

$$I^{\mathrm{A}}(k_1;\tau)I^{\mathrm{A}}(k_2;\tau) = I^{\mathrm{A}}(k_1,k_2;\tau) + I^{\mathrm{A}}(k_2,k_1;\tau),$$
$$I^{\mathrm{A}}(k_1,k_2;\tau) = (-1)^{k_1+k_2} I^{\mathrm{A}}(k_2,k_1;\tau),$$
$$\frac{1}{2} I^{\mathrm{A}}(0;\tau) = I^{\mathrm{A}}(0,0;\tau) = \frac{1}{2} \in \mathbb{Q},$$
$$I^{\mathrm{A}}(0,5;\tau) = I^{\mathrm{A}}(2,3;\tau) = -\frac{(2\pi i)^4}{12} \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n} q^n.$$

Consider the component of $\mathcal{EZ}^{\mathrm{A}}$ of length n and weight k:

$$\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}}) := \mathrm{Span}_{\mathbb{Q}}\{I^{\mathrm{A}}(k_1, \ldots, k_l; \tau) \,|\, k_1 + \ldots + k_l = k,\ l \leq n\}.$$

Problem

Compute the dimensions

$$D_{k,n}^{\mathrm{ell}} := \dim_{\mathbb{Q}}(\mathcal{L}_n(\mathcal{EZ}_k^{\mathrm{A}})/\mathcal{L}_{n-1}(\mathcal{EZ}_k^{\mathrm{A}})).$$

Length one

The differential equation in length one yields

$$I^{\mathrm{A}}(2k;\tau) = -2\zeta(2k) = \frac{B_{2k}\cdot(2\pi i)^{2k}}{(2k)!}, \quad I^{\mathrm{A}}(2k+1;\tau) = 0.$$

Therefore, with the convention $\mathcal{L}_0(\mathcal{E}\mathcal{Z}_k^{\mathrm{A}}) = \begin{cases} \mathbb{Q} & k = 0, \\ \{0\} & \text{else} \end{cases}$

$$D_{k,1}^{\mathrm{ell}} = \begin{cases} 1 & k \geq 2 \text{ even} \\ 0 & \text{else} \end{cases}$$

Observation

The transcendence of π implies there are no $\mathbb{Q}$-linear relations in length one in different weights.

Relations in length two

Relations between elliptic multiple zeta values can be written down compactly using generating series

$$\mathcal{I}^A(X_1,\dots,X_n;\tau) := \sum_{k_1,\dots,k_n\geq 0} I^{\mathrm{A}}(k_1,\dots,k_n;\tau)X_1^{k_1-1}\dots X_n^{k_n-1}.$$

Proposition (M.)

The following functional equations hold:

$$\mathcal{I}^A(X_1,X_2;\tau)+\mathcal{I}^A(X_2,X_1;\tau)\equiv 0 \mod \mathcal{L}_1(\mathcal{EZ}^{\mathrm{A}})$$
$$\mathcal{I}^A(X_1,X_2;\tau)+\mathcal{I}^A(X_1+X_2,-X_2;\tau)+\mathcal{I}^A(-X_1-X_2,X_1;\tau)\equiv 0 \mod \mathcal{L}_1(\mathcal{EZ}^{\mathrm{A}}).$$

Comparing coefficients yields a family of $\mathbb{Q}$-linear relations, called *Fay-shuffle relations*.

Length two

Theorem (M.)

(i) $\mathcal{L}_2(\mathcal{E}\mathcal{Z}^{\mathrm{A}}) := \sum_{k\geq 0} \mathcal{L}_2(\mathcal{E}\mathcal{Z}_k^{\mathrm{A}})$ is graded for the weight:

$$\mathcal{L}_2(\mathcal{E}\mathcal{Z}^{\mathrm{A}}) := \bigoplus_{k\geq 0} \mathcal{L}_2(\mathcal{E}\mathcal{Z}_k^{\mathrm{A}}).$$

(ii)

$$D_{k,2}^{\mathrm{ell}} = \begin{cases} 0 & k \text{ even} \\ \left\lfloor \frac{k}{3} \right\rfloor + 1 & k \text{ odd} \end{cases}$$

(iii) The Fay-shuffle relations give all the $\mathbb{Q}$-linear relations between the $I^{\mathrm{A}}(k_1, k_2; \tau)$ (modulo length one).

A priori, there are $k+2$ A-elliptic multiple zeta values of weight k and length at most two. (ii) tells us there are many relations among them.

L-functions of modular forms

$$\zeta(k_1, \ldots, k_n) = \sum_{0<m_1<\ldots<m_n} \frac{1}{m_1^{k_1} \ldots m_n^{k_n}}.$$

For $n = 1$, get special values of the Riemann zeta function $\zeta(s)$ (the prime example of an L-function).

Proposition (Hecke)

Let $f = \sum_{n=0}^{\infty} a_n q^n$ be a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. The series

$$L(f; s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\Re(s) > k$, and can be extended to a meromorphic function on all of $\mathbb{C}$.

Example

For the Hecke-normalized Eisenstein series $\widetilde{E}_{2k}(\tau) := \frac{(2k-1)!}{2(2\pi i)^{2k}} E_{2k}(\tau)$, $k \geq 2$, one gets

$$L(\widetilde{E}_{2k}; s) = \zeta(s)\zeta(s - 2k + 1).$$

The constant term of the elliptic KZB associator

Let $A_\infty := \lim_{\tau\to i\infty} A(\tau)$ and $B_\infty := \lim_{\tau\to i\infty} B(\tau)$.

Theorem (Enriquez)

We have

$$A_\infty = e^{\pi i t}\Phi_{\mathrm{KZ}}(\tilde{y},t)e^{2\pi i\tilde{y}}\Phi_{\mathrm{KZ}}(\tilde{y},t)^{-1},$$
$$B_\infty = \Phi_{\mathrm{KZ}}(-\tilde{y}-t,t)e^{2\pi i a}\Phi_{\mathrm{KZ}}(\tilde{y},t)^{-1},$$

where $\tilde{y} = -\frac{\mathrm{ad}(a)}{e^{2\pi i\,\mathrm{ad}(a)}-1}(b),\ t = -[a,b]$.

The formal logarithms of A_∞, B_∞ are Lie series

$$\log(A_\infty), \log(B_\infty) \in \widehat{\mathcal{L}},$$

where $\widehat{\mathcal{L}}$ is the completion of $\mathcal{L} = \mathrm{Lie}(a,b)$ for its lower central series.

The relation to periods of Eisenstein series

Let $\widehat{\mathcal{L}}^{(1)} = [\widehat{\mathcal{L}}, \widehat{\mathcal{L}}]$, and $\widehat{\mathcal{L}}^{(2)} = [\widehat{\mathcal{L}}^{(1)}, \widehat{\mathcal{L}}^{(1)}]$.

Meta-abelian quotient of $\widehat{\mathcal{L}}$

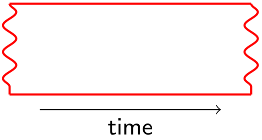
$$\widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(2)} \cong \widehat{\mathcal{L}}^{(1)}/\widehat{\mathcal{L}}^{(2)} \rtimes \widehat{\mathcal{L}}/\widehat{\mathcal{L}}^{(1)} \cong \mathbb{C}[[\mathrm{ad}(a), \mathrm{ad}(b)]] \rtimes (\mathbb{C}a \oplus \mathbb{C}b).$$

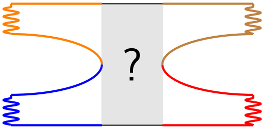
Let $\log(A_\infty)^{(1)}$, $\log(B_\infty)^{(1)}$ be the images in $\mathbb{C}[[\mathrm{ad}(a), \mathrm{ad}(b)]]$.

Theorem (M.)

The coefficients of $\log(A_\infty)^{(1)}$, $\log(B_\infty)^{(1)}$ are, up to explicit powers of $2\pi i$ and rational numbers, precisely the special values of $L(\widetilde{E}_{2k}, s)$, for $1 \leq s \leq 2k-1$ (the "periods" of $\widetilde{E}_{2k}$).

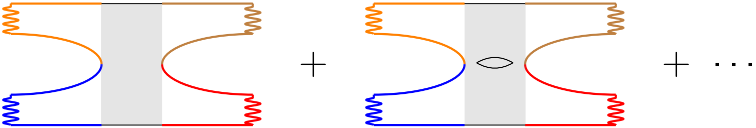
Relation to (open) string amplitudes





What happens in between?

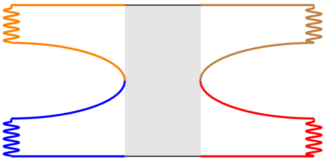
Genus expansion:



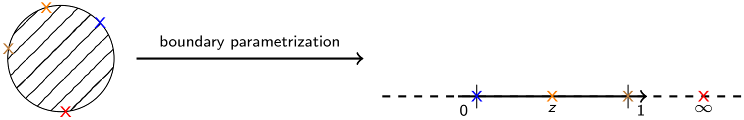
Rough mathematical description of amplitude computation

Integrate Green's functions (solutions to $\partial\overline{\partial}G \sim_{\mathbb{C}} \delta$ with some extra condition) on compact surfaces with boundaries ("worldsheets").

Genus zero



- only one compact surface with boundary: disk



(use $\mathrm{SL}_2(\mathbb{R})$-action to map three points to $0, 1, \infty$)

- Green's function $G_0(z_i, z_j) \sim_{\mathbb{C}} \log|z_i - z_j|^2$ (since $\partial\overline{\partial}\log|z_i - z_j|^2 = 2\pi\delta(z_i - z_j)$).

The four-point amplitude at genus zero

After factoring out field theory subamplitudes, the result is

$$s_{12} \int_0^1 z^{s_{12}-1}(1-z)^{s_{23}} \mathrm{d}z$$

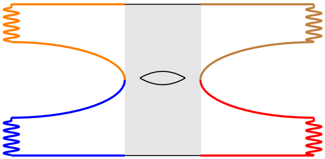
where s_{ij} are certain variables ("Mandelstam variables") carrying physical information.

Note

$$a \int_0^1 z^{a-1}(1-z)^b \mathrm{d}z = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(1+a+b)}$$
$$= \exp\left(\sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{k}\left(a^k + b^k - (a+b)^k\right)\right)$$

(Relation between string amplitudes and (single) zeta values)

Genus one



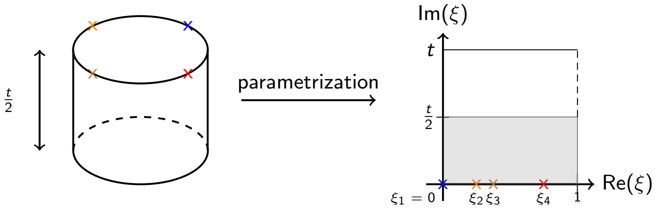
- Compact surfaces with boundary: cylinder (planar/non-planar) and Moebius strip
- Green's function is

$$G_1(\xi_i, \xi_j, \tau) = \log |\theta_\tau(\xi_i - \xi_j)|^2 - \frac{2\pi}{\mathrm{Im}(\tau)} \mathrm{Im}(\xi_i - \xi_j)^2.$$

Key observation

$\frac{\partial}{\partial \xi} G_1 \mathrm{d}\xi = \omega^{(1)}$ (relation with elliptic multiple zeta values!).

Computing the planar cylinder contribution at four points I



We are interested in computing

$$I_{1234}(s_{ij},\tau) = \iiint\limits_{0\leq\xi_2\leq\xi_3\leq\xi_4\leq 1} \exp\left(\sum_{i<j}^{4} s_{ij}\, G_1(\xi_i,\xi_j,\tau)\Big|_{\xi_1=0}\right) \mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4.$$

Result (Broedel–Mafra–M.–Schlotterer)

The coefficients of $I_{1234}(s_{ij},\tau)$ are explicitly computable A-elliptic multiple zeta values.

Key input: Fay identity for $\omega^{(k)}$ (genus one analog of partial fraction decomposition).

Computing the planar cylinder contribution at four points II

Result

To third order in the s_{ij}, we have:

$$I_{1234}(s_{ij},\tau) = I^{\mathrm{A}}(0,0,0;\tau) - 2I^{\mathrm{A}}(0,1,0,0;\tau)(s_{12}+s_{23}) + 2I^{\mathrm{A}}(0,1,1,0,0;\tau)(s_{12}^2+s_{23}^2) - 2I^{\mathrm{A}}(0,1,0,1,0;\tau)s_{12}s_{23} + \ldots$$

Summary of main results

1. Elliptic multiple zeta values are coefficients of the elliptic KZB associator (regularized monodromy of the elliptic KZB connection).
2. They can be decomposed uniquely into iterated Eisenstein integrals. The occurring linear combinations of iterated Eisenstein integrals are controlled by a special Lie algebra of derivations, and ultimately by modular forms for $\mathrm{SL}_2(\mathbb{Z})$.
3. They satisfy interesting relations (Fay-shuffle relations), which can be determined completely in some special cases.
4. Their constant terms generalize special values of L-functions of Eisenstein series.
5. Elliptic multiple zeta values appear in computations in string theory.

THANK YOU!

$$I^{\mathrm{A}}(0,3,5;\tau) = \frac{\pi^6}{63}\left(\sum_{0<n}\frac{2\sigma_3(n)+7\sigma_5(n)-9\sigma_9(n)}{n^2}q^n + 1680\sum_{0<m<n}\frac{\sigma_5(m)\sigma_3(n-m)}{mn}q^n\right)$$